Momentum Conservation (Tipler-Mosca Chapter 8)

The Center of Mass (CM):

The CM $\vec{r}_{\rm cm}$ moves as if all the external foces acting on the system were acting on the total mass M of the system located at this point. In particular, it moves with constant velocity, if the external forces acting on the system add to zero.

Definition:

$$M\,\vec{r}_{\rm cm} = \sum_{i=1}^{n} m_i\,\vec{r}_i \quad \text{where} \quad M = \sum_{i=1}^{n} m_i\ .$$

Here the sum is over the particles of the system, m_i are the masses and $\vec{r}_i$ are the position vectors of the particles. In case of a continuous object, this becomes

$$M\,\vec{r}_{\rm cm} = \int \vec{r}\,dm$$

where dm is the position element of mass located at position $\vec{r}$, see figure 8-4 of Tipler-Mosca.

Example: Figures 8-1, 8-2 and 8-3 of Tipler-Mosca.

Two masses a placed on the x-axis and $x_{\rm cm}$ is defined by

$$M\, x_{\rm cm} = m_1\, x_1 + m_2\, x_2 \quad \text{where} \quad M = m_1 + m_2 \ .$$

If the masses are equal (8-1), the CM is midway between them.

If the masses are unequal (8-2), the CM is closer to the heavier mass.

Let us choose (8-3) $x_1 = 0$. Then

$$x_{\rm cm} = \frac{m_2}{M}\, x_2 = \frac{m_2}{m_1 + m_2}\, x_2 \ .$$

PRS: Assume a 8-kg mass is at the origin and a 4-kg mass at x=0.6 m. The CM is then at (pick one):

1. $7.2\, m$ 2. $0.3\, m$ 3. $0.2\, m$ 4. $0.4\, m$ 5. $0.1\, m$ 6. $1.8\, m$

Gravitational Potential Energy and CM

The gravitational potential energy of a system of particles is given by

$$U = \sum_{i=1}^{n} m_i \, g \, h_i = g \sum_{i=1}^{n} m_i \, h_i$$

Therefore, by definition of the CM

$$U = g \, M \, h_{\rm cm} \quad \text{with} \quad M \, h_{\rm cm} = \sum_{i=1}^{n} m_i \, h_i$$

We can use this result to locate the CM experimentally: If we suspend an irregular object from a pivot, the object will rotate until the CM reaches its lowest point and the CM lies somewhere on the vertical line drawn directly downward from the pivot. By repeating this for several pivots, we find the CM.

Demonstration: CM of Florida (compare figure 8-11 of Tipler-Mosca).

For the mathematicaly ambitious:
Finding the CM by Integration

Example 1: Uniform Stick Figure 8-12 of Tipler-Mosca.

$$dm = M\,\frac{dx}{L} = \frac{M}{L}\,dx$$

$$M\,x_{\rm cm} = \int x\,dm = \frac{M}{L}\int_0^L x\,dx = \frac{M}{2L}\,x^2\bigg|_0^L = \frac{M\,L}{2}$$

$$x_{\rm cm} = \frac{L}{2}\;.$$

Example 2: Semicircular Hoop Figure 8-13 of Tipler-Mosca.

$$dm = M\,\frac{ds}{\pi\,R} = \frac{M}{\pi\,R}\,R\,d\theta$$

x coordinate: $x_{\rm cm} = 0$

$$M\, x_{\rm cm} = \int x\, dm = \frac{M}{\pi\, R} \int_0^{\pi} x\, R\, d\theta = \frac{M}{\pi\, R} \int_0^{\pi} R\, \cos(\theta)\, R\, d\theta$$

$$= \frac{M\, R}{\pi} \int_0^{\pi} \cos(\theta)\, d\theta = \frac{M\, R}{\pi} \sin(\theta)\Big|_0^{\pi} = 0\ .$$

y coordinate: $y_{\rm cm} = 2\, R/\pi$

$$M\, y_{\rm cm} = \int y\, dm = \frac{M}{\pi\, R} \int_0^{\pi} y\, R\, d\theta = \frac{M}{\pi\, R} \int_0^{\pi} R\, \sin(\theta)\, R\, d\theta$$

$$= \frac{2\, M\, R}{\pi} \int_0^{\pi/2} \sin(\theta)\, d\theta = -\frac{2\, M\, R}{\pi} \cos(\theta)\Big|_0^{\pi/2} = \frac{2 M\, R}{\pi}$$

Motion of the CM

Velocity of the CM:

$$M \frac{d\vec{r}_{\rm cm}}{dt} = m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \ldots = \sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt} \ .$$

Differentiating again give the acceleration of the CM:

$$M \vec{a}_{\rm cm} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots = \sum_{i=1}^{n} m_i \vec{a}_i$$

The acceleration of each particle is due to internal and external forces:

$$m_i \vec{a}_i = \vec{F}_i = \vec{F}_{i,\rm int} + \vec{F}_{i,\rm ext}$$

According to Newton's third law the internal forces come in pairs which cancel and we find:

$$M\,\vec{a}_{\mathrm{cm}} = \sum_{i=1}^{n} \vec{F}_{i,\mathrm{ext}} = \vec{F}_{\mathrm{net,ext}}$$

The CM moves like a particle of mass $M = \sum_i m_i$ under the influence of the net external force acting on the system.

Example: Tipler-Mosca figure 8-18.

80 kg and 120 kg sit on a 60 kg rowboat 2 m apart. The boat is at rest on a calm lake and 80 kg sits at the center of the boat.

80 g and 120 kg switch places. How far does the boat move?

Solution: As there are no external forces, the CM does not change. Therefore, the difference in the CM position before and after the move is how far the boat moves.

We choose the center of the boat as the coordinate origin and find:

Before the move

$$x_{\mathrm{cm}} = \frac{(80\,\mathrm{kg})\,(0) + (60\,\mathrm{kg})\,(0) + (120\,\mathrm{kg})\,(2\,\mathrm{m})}{80\,\mathrm{kg} + 60\,\mathrm{kg} + 120\,\mathrm{kg}} = 0.923\,\mathrm{m}$$

After the move

$$x'_{\mathrm{cm}} = \frac{(120\,\mathrm{kg})\,(0) + (60\,\mathrm{kg})\,(0) + (80\,\mathrm{kg})\,(2\,\mathrm{m})}{80\,\mathrm{kg} + 60\,\mathrm{kg} + 120\,\mathrm{kg}} = 0.615\,\mathrm{m}$$

Therefore, the boat moves

$$\triangle x = x_{\mathrm{cm}} - x'_{\mathrm{cm}} = 0.923\,\mathrm{m} - 0.615\,\mathrm{m} = 0.308\,\mathrm{m}\ .$$

Momentum Conservation

Definition: The mass of a particle times it velocity is called momentum

$$\vec{p} = m\,\vec{v}\ .$$

Newton's second law can be written as

$$\vec{F}_{\rm net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\,\frac{d\vec{v}}{dt} = m\,\vec{a}$$

as the masses of our particles have been constant.

The total momentum $\vec{P}$ of a system is the sum of the momenta of the individual particles:

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i\,\vec{v}_i = M\,\vec{v}_{\rm cm}$$

Differentiating this equation with respect to time, we obtain

$$\frac{d\vec{P}}{dt} = M\,\frac{d\vec{v}_{\rm cm}}{dt} = M\,\vec{a}_{\rm cm} = \vec{F}_{\rm net,ext}$$

The law of momentum conservation: When the net external force is zero, the total momentum is constant

$$\vec{F}_{\rm net,ext} = 0 \quad \Rightarrow \quad \vec{P} = {\rm constant}.$$

Example:

An astronaut of m_a =60 kg weight is at rest relative to her spaceship. She pushes a detached solar pannel of weight m_a =80 kg away into space at 0.3 m/s relative to the spaceship. What is her subsequent velocity relative to the spaceship?

Solution: The CM of the two body system, astronaut and pannel, stays at rest (relative to the spaceship), as now external forces are acting. Momentum conservation gives

$$0 = p_a + p_p = m_a\, v_a + m_p\, v_p$$

$$v_a = -\frac{m_p\, v_p}{m_a} = -\frac{80\,\mathrm{kg}}{60\,\mathrm{kg}}\,(0.3\,\mathrm{m/s}) = -0.4\,\mathrm{m/s}$$