The Vector Nature of Rotation

1. The angular velocity $\vec{\omega}$.

   Right-hand-rule: Tipler-Mosca figure 10-2.

2. In accordance with the right-hand-rule the torque is defined as a vector: Figures 10-3 and 10-4 of Tipler-Mosca.

   Mathematically this is expressed by defining the torque as cross (or vector) product:

   $$\vec{\tau} = \vec{r} \times \vec{F}$$

   General definition of the vector product:

   $$\vec{C} = \vec{A} \times \vec{B} = A\,B\,\sin(\phi)\,\hat{n}$$

   where ϕ is the angle between the vectors and $\hat{n}$ is a unit vector that is perpendicular to $\vec{A}$ and $\vec{B}$ and in the direction given by the right-hand-rule: Tipler-Mosca figures 10-5 and 10-6.

   Note that the magnitude C of $\vec{C}$ is the area of the parallelogram.

Properties of the cross product:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad \Rightarrow \quad \vec{A} \times \vec{A} = 0$$

Distributive law:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Product rule for derivatives:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

Vector products of the unit vectors are related by cyclic permutation:

$$\hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y} \ .$$

In Tipler notation:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j} \ .$$

Angular Momentum

Definition:

$$\vec{L} = \vec{r} \times \vec{p}$$

Like torque angular momentum is defined with respect to the point in space where the position vector $\vec{r}$ originates. For a rotation around a symmetry axis $\widehat{z}$ we find

$$\vec{L} = I\,\vec{\omega}$$

see figures 10-10 and 10-11 of Tipler-Mosca.

Proof:

$$\vec{L} = R\,m\,v_1\,\hat{z} + R\,m\,v_2\,\hat{z} + r_z\,m\,v_1\,\hat{n} - r_z\,m\,v_2\,\hat{n}$$

If $\hat{z}$ is a symmetry axis $v_1 = v_2$ and:

$$\vec{L} = 2\,m\,R\,v\,\hat{z} = 2\,m\,R^2\,\vec{\omega} = I\,\vec{\omega}\ .$$

The angular momentum about any point O' is the angular momentum about the center of mass, called spin angular momentum, plus the angular momentum associated with the motion of the center of mass about O', called orbital angular momentum:

$$\vec{L} = \vec{L}_{\rm orbit} + \vec{L}_{\rm spin} = \vec{r}_{\rm cm} \times M\,\vec{v}_{\rm cm} + \sum_i \vec{r'}_i \times m_i\,\vec{u}_i$$

Example: Earth orbitting around the sun, figure 10-15 of Tipler-Mosca.

Torque and Angular Momentum

The net external torque acting on a system equals the rate of change of the angular momentum of the system:

$$\sum_i \vec{\tau}_{i,\mathrm{ext}} = \frac{d\vec{L}}{dt}$$

Proof:

$$\vec{\tau}_{\mathrm{net}} = \vec{r} \times \vec{F}_{\mathrm{net}} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

and

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\,\vec{v} = 0\ .$$

Conservation of Angular Momentum

If the net external torque acting on a system is zero, the total angular momentum of the system is constant.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm net} = 0 \quad \Rightarrow \quad \vec{L} = \text{constant.}$$

This has important consequences!

PRS:

Rotation on a chair. By pulling the weights closer to the body, the rotation will become (pick one):

1. slower   2. faster

Another Demonstration: Gyroscope.

Precession of a Gyroscope

We have (Tipler-Mosca figures 10-20 and 10-21)

$$\vec{\tau}_{\rm net} = \frac{d\vec{L}}{dt} \quad \text{or} \quad \vec{\tau}_{\rm net}\, dt = d\vec{L}$$

and for the net torque

$$\vec{\tau}_{\rm net} = \vec{r}(t) \times M\, \vec{g}\ .$$

Now,

$$d\vec{L} = L\, d\hat{l} + \hat{l}\, d|\vec{L}|$$

where $|\vec{L}|$ is the magnitude of the angular momentum and $\hat{l}$ the unit vector in the direction of the momentum. Note, Tipler-Mosca on p.317 is using dL for the magnitude $|d\vec{L}|$ and not for the infinitesimal change of the magnitude $|\vec{L}|$. From the derivation of Tipler it remains unclear where the approximation is.

The angular momentum of the wheel around its CM symmetry (spin) axis is

$$\vec{L}_{\mathrm{cm}} = I_s \, \vec{\omega}_s \ .$$

Assume that $\vec{L}_{\mathrm{cm}}$ is very large compared to:

1. The orbital angular momentum.

2. The change (derivative) of the magnitude of the angular momentum.

Then, we can approximate the change in the angular momentum by

$$\vec{\tau}_{\mathrm{net}} \, dt = d\vec{L} = I_s \, \omega_s \, d\hat{r}$$

where $\hat{r}$ is the unit vector in the direction of the symmetry axis.

This can be written as

$$D\,M\,g\,\hat{\tau}\,dt = I_s\,\omega_s\,\omega_p\,\hat{\tau}\,dt$$

where D is the distance of the wheel from the center, $\hat{\tau}$ the unit vector in the direction of the torque and ω_p the angular velocity of the unit vector $\hat{r}$ around the origin. (Note,

$$d\vec{r} = \vec{v}\,dt = r\,\vec{\omega}\,dt$$

and $r = 1$ for the unit vector.)

Due to the angular velocity

$$\omega_p = \frac{M\,g\,D}{I_s\,\omega_s}$$

we have a motion in the direction of the torque, which is called precession.

There are corrections to our approximation, in particuar due to the initial gain of orbital angular momentum. These corrections lead to an up-and-down oscillation, called nutation, of the axle.

Clutch

Tipler-Mosca figure 10-25: A disk is rotating with an initial angular speed ω_1 about a frictionless symmetry axis. Its moment of inertia about this axis is I_1. It is dropped on another disk of moment of inertia I_2 about the same symmetry axis, which is initially at rest. Due to friction the two disks attain eventually a common angular speed ω_f. Find ω_f.

Angular momentum conservations gives:

$$L_f = (I_1 + I_2)\,\omega_f = L_i = I_1\,\omega_1$$

$$\omega_f = \frac{I_1\,\omega_1}{I_1 + I_2}$$

How much kinetic energy is lost?

$$K_f = (I_1 + I_2)\,\frac{\omega_f^2}{2} \quad \text{and} \quad K_i = I_1\,\frac{\omega_1^2}{2}$$

$$\frac{K_f}{K_i} = \frac{I_1}{I_1 + I_2}$$

Merry-Go-Round

Tipler-Mosca figure 10-26: A merry-go-round of radius $R = 2\,$m and moment of inertia $I_m = 500\,$kg·m^2 is rotating about a frictionless pivot, making one revolution per $5\,$s. A child of mass $m = 25\,$kg, originally standing at the center walks out the rim. Consider the child as a point particle of mass m and find the new angular speed of the merry-go-round.

At the rim:

$$I_{\rm child} = m\,R^2$$

Angular momentum conservation:

$$L_f = (I_m + m\,R^2)\,\omega_f = L_i = I_m\,\omega_i$$

(Initially $I_{\rm child} = 0$ as it stays at the center.) Therefore,

$$\omega_f = \frac{I_m\,\omega_i}{I_m + m\,R^2} = \frac{500}{500 + 25 \cdot 2^2}\,\frac{1\,\rm rev}{5\,\rm s} = \frac{1\,\rm rev}{6\,\rm s}$$