Oscillations (Chapter 14)

Oscillations occur when a system is disturbed from stable equilibrium. Examples: Water waves, clock pendulum, string on musical instruments, sound waves, electric currents, ...

Simple Harmonic Motion

Example: Hooke's law for a spring.

$$F_x = m\,a = -k\,x = m\,\frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$

The acceleration is proportional to the displacement and is oppositely directed. This defines harmonic motion.

The time it takes to make a complete oscillation is called the period T. The reciprocal of the period is the frequency

$$f = \frac{1}{T}$$

The unit of frequency is the inverse second s^{-1}, which is called a hertz Hz.

Solution of the differential equation:

$$x = x(t) = A\,\cos(\omega\,t + \delta) = A\,\sin(\omega\,t + \delta - \pi/2)$$

A, ω and δ are constants: A is the amplitude, ω the angular frequencey, and δ the phase.

$$v = v(t) = \frac{dx}{dt} = -\omega\,A\,\sin(\omega\,t + \delta)$$

$$a = a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2\,A\,\cos(\omega\,t + \delta) = -\omega^2\,x$$

Therefore, for the spring

$$\omega = \sqrt{\frac{k}{m}} \; .$$

Initial conditions: The amplitude A and the phase δ are determined by the initial position x_0 and initial velocity v_0:

$$x_0 = A \cos(\delta) \;\; \text{and} \;\; v_0 = -\omega A \sin(\delta) \; .$$

In particular, for the initial position $x_0 = x_{\max} = A$, the maximum displacement, we have $\delta = 0 \Rightarrow v_0 = 0$.

The period T is the time after which x repeats:

$$x(t) = x(t+T) \Rightarrow \cos(\omega\, t + \delta) = \cos(\omega\, t + \omega\, T + \delta)$$

Therefore,

$$\omega\, T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi\, f$$

is the relationship between the frequency and the angular frequency. For Hooke's law:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Simple Harmonic and Circular Motion

Imagine a particle moving with constant speed v in a circle of radius $R = A$. Its angular displacement is

$$\theta = \omega\, t + \delta \;\; \text{with} \;\; \omega = \frac{v}{R}\,.$$

The x component of the particle's position is (figure 14-6 of Tipler-Mosca)

$$x = A\, \cos(\theta) = A\, \cos(\omega\, t + \delta)$$

which is the same as for simple harmonic motion.

Demonstration: Projected shadow of a rotating peg an an object on a spring move in unison when the periods agree.

Energy in Simple Harmonic Motion

When an objects undergoes simple harmonic motion, the systems's potential and kinetic energies vary in time. Their sum, the total energy $E = K + U$ is constant. For the force $-k\,x$, with the convention $U(x = 0) = 0$, the system's potential energy is

$$U = -\int_0^x F(x')\,dx' = \int_0^x k\,x'\,dx' = \frac{k}{2}\,x^2 \ .$$

Substitution for simple harmonic motion gives

$$U = \frac{k}{2}\,A^2\,\cos^2(\omega\,t + \delta) \ .$$

The kinetic energy is

$$K = m\,\frac{v^2}{2}$$

Substitution for simple harmonic motion gives

$$K = \frac{1}{2} m\,\omega^2\,A^2\,\sin^2(\omega\,t + \delta)^2\ .$$

Using $\omega^2 = k/m$,

$$K = \frac{k}{2}\,A^2\,\sin^2(\omega\,t + \delta)^2\ .$$

The total energy is the sum

$$E = U + K = \frac{k}{2}\,A^2\,\left[\cos^2(\omega\,t + \delta) + \sin^2(\omega\,t + \delta)^2\right] = \frac{k}{2}\,A^2\ .$$

I.e., the total energy is proportional to the amplitude squared.

Plots of U and K versus t: Figures 14-7 of Tipler-Mosca.

Potential energy as function of x: Figure 14-8 of Tipler-Mosca.

Average kinetic and potential energies:

$$U_{\mathrm{av}} = K_{\mathrm{av}} = \frac{1}{2}\,E_{\mathrm{total}}\ .$$

Turning points at the maximum displacement $|x| = A$.

PRS: At the turning points the total energy is?

1. All kinetic. 2. All potential.

3. Half potential and half kinetic.

At $x = 0$ the total energy is?

1. Kinetic. 2. Potential.

3. Half potential and half kinetic.

General Motion Near Equilibrium

Any smooth potential curve $U(x)$ that has a minimum at, say x_1, can be approximated by

$$U = A + B\left(x - x_1\right)^2$$

and the force is given by

$$F_x = -\frac{dU}{dx} = -2B\left(x - x_1\right) = -k\left(x - x_1\right)$$

with $k = 2B$.

Compare figures 14-9 and 14-10 of Tipler-Mosca.

Examples of Oscillating Systems

Object on a Vertical Spring:

$$m \frac{d^2y}{dt^2} = F_y(y) = -k\,y + m\,g \ .$$

Equilibrium position:

$$0 = F_y(y_0) = -k\,y_0 + m\,g \ \Rightarrow \ y_0 = m\,g/k \ .$$

Substitution of $y = y' + y_0$ into Newton's equation gives

$$m \frac{d^2(y' + y_0)}{dt^2} = m \frac{d^2y'}{dt^2} = -k\,y' - k\,y_0 + m\,g = -k\,y' \ .$$

This is the equation of harmonic motion with the solution

$$y' = A \cos(\omega\,t + \delta) \ .$$

So, if we measure the displacement from the equilibrium position, we can forget about the effect of gravity (figure 14-11 of Tipler-Mosca).

The Simple Pendulum: Figure 14-14 of Tipler-Mosca.

$$s = L\,\phi \quad \text{where} \quad \phi \quad \text{is in radians.}$$

Newton's second law:

$$F_t = m\,\frac{d^2s}{dt^2} = m\,L\,\frac{d^2\phi}{dt^2}\ .$$

Question (PRS): The absolute value of the tangential force is

$$1.\quad |F_t| = m\,g\,\sin(\phi)\ . \qquad\qquad 2.\quad |F_t| = m\,g\,\cos(\phi)\ .$$

Removing the absolute value from F_t, the sign on the right-hand-side is:

1. positive.
2. negative.

Therefore,

$$F_t = -m\,g\,\sin(\phi) = \frac{d^2 s}{dt^2} = m\,L\,\frac{d^2\phi}{dt^2}$$

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\,\sin(\phi)\ .$$

For small oscillations we have $\sin(\phi) \approx \phi$, and

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\,\phi = -\omega^2\,\phi\ .$$

The period is thus

$$T = \frac{2\pi}{\omega} = 2\pi\,\sqrt{\frac{L}{g}}$$

and the solution for the motion of the angle is

$$\phi = \phi_0\,\cos(\omega\,t + \delta)$$

where ϕ_0 is the maximum angular displacement.

Pendulum in an Accelerated Reference Frame:

Figure 14-15 of Tipler-Mosca.

The solution is found from the simple pendulum by replacing g with g' where

$$\vec{g'} = \vec{g} - \vec{a}$$

and $\vec{a}$ is the acceleration. As $\vec{g}$ and $\vec{a}$ are perpendicular, we have

$$g' = |\vec{g'}| = \sqrt{\vec{g}^{\,2} + \vec{a}^{\,2}} \ .$$

The Physical Pendulum: Figure 14-17 of Tipler-Mosca.

This is a rigid object pivoted about a point other than its center of mass. It will oscillate when displaced from equilibrium. Newton's second law of rotation is:

$$\tau = I\,\alpha = I\,\frac{d^2\phi}{dt^2}$$

where α is the angular acceleration and I the moment of inertia about the pivot point.

Question (PRS): The torque is given by

1. $\tau = -M\,g\,D\,\sin(\phi)$.
2. $\tau = -M\,g\,D\,\cos(\phi)$.

Therefore,

$$-M\,g\,D\,\sin(\phi) = I\,\frac{d^2\phi}{dt^2}$$

$$\frac{d^2\phi}{dt^2} = -\frac{M\,g\,D}{I}\,\sin(\phi) \approx -\frac{M\,g\,D}{I}\,\phi = -\omega^2\,\phi$$

and the period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{M\,g\,D}}\;.$$

Damped Oscillations

Left to itself, an oscillation stops eventually, because mechanical energy is dissipated by frictional forces. Such motion is said to be damped. The damping force can be represented by the empirical expression

$$\vec{F}_d = -b\,\vec{v}$$

where b is a constant. The motion of a damped system can be calculated from Newton's second law

$$F_x = -k\,x - b\,\frac{dx}{dt} = m\,\frac{d^2x}{dt^2}$$

The solution for this equation is found using standard methods for solving differential equations. The result is

$$x = A_0\,e^{-(b/2m)t}\,\cos(\omega'\,t + \delta) = A_0\,e^{-t/2\tau}\,\cos(\omega'\,t + \delta)$$

where A_0 is the maximum amplitude and

$$\tau = \frac{m}{b}$$

is called decay time or time constant.

The frequency ω' is given by

$$\omega' = \omega_0 \sqrt{1 - \left(\frac{b}{2m\,\omega_0}\right)^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Here ω_0 is the frequency without damping. The dashed curves in figure 14-20 of Tipler-Mosca correspond to $x = \pm A$ where A is given by

$$A = A_0\, e^{-(b/2m)t} = A_0\, e^{-t/\tau} \ .$$

If the damping constant b is gradually increased, we have

$$\omega' = 0 \ \text{ at the critical value } \ b_c = 2m\,\omega_o \ .$$

When $b \geq b_c$, the system does not oscillate:

$b = b_c$: The system is critically damped.

$b > b_c$: The system is overdamped.

One uses critical damping to return to equilibrium quickly. Example: Shock absorbers of a car.

$b < b_c$: The system is underdamped (often simply called damped).

Energy of an underdamped oscillator:

$$E = \frac{1}{2} m \, \omega^2 \, A^2 = \frac{1}{2} m \, \omega^2 \, \left(A_0 \, e^{-t/2\tau} \right)^2 = E_0 \, e^{-t/\tau}$$

$$\text{where} \quad E_0 = \frac{1}{2} m \, \omega^2 \, A_0^2$$

The Q factor (for quality factor) of an oscillator relates to the fractional energy loss per cycle. The infinitesimal change of the energy is

$$dE = -\frac{1}{\tau} E_0 \, e^{-t/\tau} \, dt = -\frac{1}{\tau} E \, dt \ .$$

If the energy loss per period, $\triangle E$, is small, we can replace dE by $\triangle E$ and dt by T (also $\omega' \approx \omega_o$):

$$\frac{|\triangle E|}{E} = \frac{T}{\tau} = \frac{2\pi}{\omega_0 \, \tau} = \frac{2\pi}{Q}$$

with the Q factor given by

$$Q = \omega_o \, \tau = \frac{\omega_o \, m}{b} = \frac{2\pi}{(|\triangle E|/E)_{\rm cycle}} \ .$$

Driven Oscillations and Resonance

To keep a damped oscillator going, energy must be put into the system. Example: When you keep a swing going, you drive an oscillator.

The natural frequency ω_0 of an oscillator is the frequency when no driving or damping forces are present.

We assume that the oscillator is driven by a periodic motion of driving frequency ω.

When the driving frequency equals the natural frequency, the energy absorbed is at its maximum. Therefore, the natural frequency is also called the resonance frequency of the system.

In most applications the angular frequency $\omega = 2\pi f$, instead of the frequency, is used, because this is mathematically more convenient. In verbal descriptions the word "angular" is often omitted.

A resonance curve shows the average power delivered to an oscillator as function of the driving frequency: For two different damping constants the resonance curve is plotted in figure 14-24 of Tipler-Mosca. The resonance is sharp for small damping.

For small damping the ratio of the width of the resonance to the frequency can be shown to be equal to the reciprocal Q factor

$$\frac{\triangle\omega}{\omega_0} = \frac{\triangle f}{f} = \frac{1}{Q} \ .$$

Intuitively, we know how to drive an oscillator at its resonance frequency (swing, etc.). The differential equation is

$$m\,\frac{d^2x}{d\,t^2} + b\,\frac{dx}{dt} + m\,\omega_0^2\,x = F_0\,\cos(\omega t) = F_{\rm ext} \ .$$