SOLUTIONS MIDTERM PHY 4936 (Fall 2011)

PROBLEM 3

3.1 Principle of least action: Every mechanical system is characterized by a definite function $L = L(q_1, \dots, q_s,\, \dot{q}_1, \dots, \dot{q}_s,\, t)$ and the motion of the system is such that the system moves between two fixed positions at different times t_1 and t_2 in a way that for sufficiently short time differences the integral

$$S \;=\; \int_{t_1}^{t_2} L\, dt$$

takes the least possible value.

3.2 Deviation of the Euler-Lagrange equations from the least action principle in general coordinates $q_i, \dot{q}_i,\; i = 1, \dots s$:

$$\begin{aligned}
0 \;&=\; \delta \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t) \;=\; \int_{t_1}^{t_2} dt\, \left\{ L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) - L(q_i, \dot{q}_i, t) \right\} \\
&=\; \int_{t_1}^{t_2} dt\, \left\{ L(q_i, \dot{q}_i, t) + \sum_i \frac{\partial L}{\partial q_i}\, \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\, \delta \dot{q}_i - L(q_i, \dot{q}_i, t) \right\} \\
&=\; \int_{t_1}^{t_2} dt\, \left\{ \sum_i \frac{\partial L}{\partial q_i}\, \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\, \frac{d}{dt} \delta q_i \right\} \\
&=\; \int_{t_1}^{t_2} dt\, \sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right\} \delta q_i + \left[\frac{\partial L}{\partial \dot{q}_i}\, \delta q_i \right]_{t_1}^{t_2} \;=\; \int_{t_1}^{t_2} dt\, \sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right\} \delta q_i\ .
\end{aligned}$$

The last equality holds because of $\delta q_i(t_1) = \delta q_i(t_2) = 0$. As the variations are independent, the finally obtained relation is equivalent to

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \;=\; 0 \;\; \text{for} \;\; i = 1, \dots, s\ .$$

3.3 $L(q_k + \epsilon_k, \dot{q}_k, t) = L(q_k, \dot{q}_k, t)$ implies

$$0 = \frac{\partial L}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} \;\Rightarrow\; p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ conserved (generalized momentum).}$$