SOLUTIONS FINAL – PHY 4936 Fall 2011

PROBLEM 1

(1) The momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

(2) Eliminating $\dot{x}$ in favor of p we obtain the Hamiltonian

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

(3) Hamilton's equations are

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \quad \text{and} \quad \frac{\partial H}{\partial x} = kx = -\dot{p}$$

(4) Newton's force law follows

$$kx = -\dot{p} = -m\ddot{x} \quad \text{or} \quad m\ddot{x} = -kx$$

PROBLEM 2

We consider an infinitesimal rotation by an angle $\delta\vec{\phi}$ (Landau-Lifshitz, §9, p.18):

$$\delta\vec{r}_j \; = \; \delta\vec{\phi} \times \vec{r}_j \,, \quad \delta\dot{\vec{r}}_j \; = \; \delta\vec{\phi} \times \dot{\vec{r}}_j \,. \tag{1}$$

For example, if this rotation is about the z axis $|\delta\vec{r}_j| = |\delta\phi \, r_j \, \sin(\theta)|$ holds. Let us denote the components of the rotations by δx^i_j and $\delta\dot{x}^i_j$, where $i = 1, 2, 3$ labels the coordinates and $j = 1, \dots, n$ the particles. Assuming isotropy of space, the Lagrangian is invariant under an infinitesimal rotation

$$0 \; = \; \sum_j \left\{ \sum_i \frac{\partial L}{\partial x^i_j} \, \delta x^i_j + \sum_i \frac{\partial L}{\partial \dot{x}^i_j} \, \delta\dot{x}^i_j \right\} .$$

Using the definition of the generalized momentum and Euler-Lagrange equations, this reads

$$0 \; = \; \sum_j \left\{ \sum_i \dot{p}^i_j \, \delta x^i_j + \sum_i p^i_j \, \delta\dot{x}^i_j \right\} \; = \; \sum_j \left\{ \dot{\vec{p}}_j \cdot \delta\vec{r}_j + \vec{p}_j \cdot \delta\dot{\vec{r}}_j \right\} .$$

Inserting (1)

$$0 \; = \; \sum_j \left\{ \dot{\vec{p}}_j \cdot (\delta\vec{\phi} \times \vec{r}_j) + \vec{p}_j \cdot (\delta\vec{\phi} \times \dot{\vec{r}}_j) \right\}$$

and we pull out the $\delta\vec{\phi}$, which is the same for all particles:

$$0 \; = \; \sum_j \left\{ \delta\vec{\phi} \cdot (\vec{r}_j \times \dot{p}_j)) + \delta\vec{\phi} \cdot (\dot{\vec{r}}_j \times \vec{p}_j) \right\} \; = \; \delta\vec{\phi} \, \frac{d}{dt} \sum_j (\vec{r}_j \times \vec{p}_j)$$

$$\Leftrightarrow \; \sum_j (\vec{r}_j \times \vec{p}_j) \; = \; \vec{L} \; = \; \text{Constant} \,.$$

PROBLEM 3: See Homework, Problem 19.

PROBLEM 4: Compare Homework, Problem 29.

1. Given a coordinate system (x', y') which rotates with the disk, the location of the CM is $\bar{x}_{CM} = 0$ and

$$\begin{aligned}\bar{y}_{CM} &= \frac{\rho}{M}\left\{\int_0^R drr \int_0^\pi d\theta r \sin\theta + 2\int_0^R drr \int_\pi^{2\pi} d\theta r \sin\theta\right\} \\ &= \frac{\rho}{M}\left\{\frac{R^3}{3}2 - \frac{R^3}{3}4\right\} = -\frac{2}{3}\frac{\rho R^3}{M} = -\frac{4R}{9\pi} \quad \text{as} \quad M = \frac{3\pi\,\rho\,R^2}{2}\,.\end{aligned}$$

2. The relationship between the lab coordinates and the CM coordinates is

$$\begin{aligned}x_{CM} &= R\theta - |\bar{y}_{CM}|\sin\theta = R\theta - (4R/9\pi)\,\sin\theta\,, \\ y_{CM} &= R - |\bar{y}_{CM}|\cos\theta = R - (4R/9\pi)\,\cos\theta\,.\end{aligned}$$

3. To find I_3^{CM} we calculate first I_3^0 with respect to the center of disk and use the parallel axis theorem, (32.12) of Landau and Lifshitz,

$$\begin{aligned}I_3\Big|_0 &= \rho\int_0^R drr\int_0^\pi d\theta(x^2+y^2) + 2\rho\int_0^R drr\int_\pi^{2\pi} d\theta(x^2+y^2) \\ &= \rho\frac{R^4}{4}\pi + 2\rho\frac{R^4}{4}\pi = \rho R^4\frac{3}{4}\pi = \frac{1}{2}MR^2 \\ I_3\Big|_{CM} &= I_3|_0 - M\bar{y}_{CM}^2 = \frac{1}{2}MR^2 - M\frac{16}{81}\frac{R^2}{\pi^2} = \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right].\end{aligned}$$

4. The Lagrangian of the disk is

$$L = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_3\dot{\theta}^2 - M\,g\,y_{CM}\,.$$