Solution for assignment 8: Double Pendulum.

Derivation of the Lagrangian:

$$
\begin{aligned}
x_1 &= l \sin\phi \\
y_1 &= l \cos\phi \\
x_2 &= l \sin\phi + l \sin\psi \\
y_2 &= l \cos\phi + l \cos\psi \\
L &= \frac{1}{2} m l \left(\dot{x_1}^2 + \dot{y_1}^2 \right) + \frac{1}{2} m l \left(\dot{x_2}^2 + \dot{y_2}^2 \right) - m g l \left(z_1 + z_2 \right) + \text{const} \\
&= \frac{1}{2} m l \dot{\phi}^2 + \frac{1}{2} m l \left(\dot{\phi}^2 + \dot{\psi}^2 + 2 \dot{\phi}\dot{\psi} \cos(\phi - \psi) \right) + m g l \left(2 \cos\phi + \cos\psi \right) \\
&= m l \left(\dot{\phi}^2 + \frac{1}{2} \dot{\psi}^2 + \dot{\phi}\dot{\psi} \cos(\phi - \psi) \right) + m g l \left(2 \cos\phi + \cos\psi \right) \; .
\end{aligned}
$$

Taylor expansion to second order in ϕ, ψ, $\dot{\phi}$ and $\dot{\psi}$ gives

$$
L = m l \left(\dot{\phi}^2 + \frac{1}{2} \dot{\psi}^2 + \dot{\phi}\dot{\psi} \right) - m g l \left(\phi^2 + \frac{1}{2} \psi^2 \right) + 3 m g l \; .
$$

The Euler-Lagrange equation are then

$$
\begin{aligned}
0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 2 \ddot{\phi} + \ddot{\psi} + 2 \frac{g}{l} \phi \\
0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} = \ddot{\psi} + \ddot{\phi} + \frac{g}{l} \psi
\end{aligned}
$$

and in matrix notation

$$
\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} + \begin{pmatrix} 2g/l & 0 \\ 0 & g/l \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \; .
$$