Solution for assignment 13:

$$
\begin{aligned}
(d\vec{r})^2 &= (dx)^2 + (dy)^2 + (dz)^2 && (1)\\
&= (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2 && (2)\\
&= (dr)^2 + r^2 \sin^2\theta\,(d\phi)^2 + r^2 (d\theta)^2 && (3)\\
(\vec{v})^2 &= (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 && (4)\\
&= (\dot{\rho})^2 + \rho^2 (\dot{\phi})^2 + (\dot{z})^2 && (5)\\
&= (\dot{r})^2 + r^2 \sin^2\theta\,(\dot{\phi})^2 + r^2 (\dot{\theta})^2 && (6)
\end{aligned}
$$

Locally orthonormal unit vectors are given by

$$
\begin{aligned}
\hat{\rho} &= \cos(\phi)\,\hat{x} + \sin(\phi)\,\hat{y} && (7)\\
\hat{\phi} &= -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y} && (8)\\
\hat{r} &= \sin(\theta)\,\hat{\rho} + \cos(\theta)\,\hat{z} \;=\; \sin(\theta)\,\cos(\phi)\,\hat{x} + \sin(\theta)\,\sin(\phi)\,\hat{y} + \cos(\theta)\,\hat{z} && (9)\\
\hat{\theta} &= \cos(\theta)\,\hat{\rho} - \sin(\theta)\,\hat{z} \;=\; \cos(\theta)\,\cos(\phi)\,\hat{x} + \cos(\theta)\,\sin(\phi)\,\hat{y} - \sin(\theta)\,\hat{z} && (10)
\end{aligned}
$$

Usefull is table 1 of dot products between the Cartesian and the new, local orthonormal unit vectors.

	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\hat{\rho}$	$\cos\phi$	$\sin\phi$	0
$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0
$\hat{r}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\hat{\theta}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$

Table 1: Dot products between Cartesian and local orthonormal unit vectors.

The velocity vector is in Cartesian, cylindrical and spherical coordinates

$$
\begin{aligned}
\vec{v} &= \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z} && (11)\\
&= \dot{\rho}\,\hat{\rho} + \rho\,\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z} && (12)\\
&= \dot{r}\,\hat{r} + r\,\sin(\theta)\,\dot{\phi}\,\hat{\phi} + r\,\dot{\theta}\,\hat{\theta} && (13)
\end{aligned}
$$

Application (Landau-Lifshitz, p.21, Problems 1 and 2): Cartesian angular momentum components and angular momentum squared in cylindrical and spherical coordinates.

Vector products between cylindrical unit vectors:

$$\hat{\rho} \times \hat{\phi} \;=\; \hat{z}\,, \quad \hat{\phi} \times \hat{z} \;=\; \hat{\rho}\,, \quad \hat{z} \times \hat{\rho} \;=\; \hat{\phi} \tag{14}$$

Angular momentum in cylindrical coordinates:

$$\vec{M} = m\,\vec{r} \times \vec{v} \;=\; m\,(\rho\,\hat{\rho} + z\,\hat{z}) \times \left(\dot{\rho}\,\hat{\rho} + \rho\,\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}\right) \tag{15}$$

$$=\; m\left(\rho^2\dot{\phi}\,\hat{z} - \rho\,\dot{z}\,\hat{\phi} + z\,\dot{\rho}\,\hat{\phi} - z\,\rho\,\dot{\phi}\,\hat{\rho}\right) \tag{16}$$

$$=\; m\left(-z\,\rho\,\dot{\phi}\,\hat{\rho} + (z\,\dot{\rho} - \rho\,\dot{z})\;\hat{\phi} + \rho^2\dot{\phi}\,\hat{z}\right) \tag{17}$$

Therefore,

$$\vec{M}^{\,2} = M^2 \;=\; m^2\rho^2\dot{\phi}^2\left(\rho^2 + z^2\right) + m^2\,(z\,\dot{\rho} - \rho\,\dot{z})^2 \tag{18}$$

Vector products between spherical unit vectors:

$$\hat{\phi} \times \hat{r} \;=\; \hat{\theta}\,, \quad \hat{r} \times \hat{\theta} \;=\; \hat{\phi}\,, \quad \hat{\theta} \times \hat{\phi} \;=\; \hat{r} \tag{19}$$

Angular momentum in spherical coordinates:

$$\vec{M} = m\,\vec{r} \times \vec{v} \;=\; m\,r\,\hat{r} \times \left(\dot{r}\,\hat{r} + r\,\sin(\theta)\,\dot{\phi}\,\hat{\phi} + r\,\dot{\theta}\,\hat{\theta}\right) \tag{20}$$

$$=\; m\,r^2\left(\dot{\theta}\,\hat{\phi} - \sin(\theta)\,\dot{\phi}\,\hat{\theta}\right) \tag{21}$$

Therefore,

$$\vec{M}^{\,2} = M^2 \;=\; m^2\,r^4\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \tag{22}$$

Now use $M_x = \vec{M} \cdot \hat{x}$ and so on.

Cylindrical coordinates:

$$M_x \;=\; \vec{M} \cdot \hat{x} \;=\; m\,(\rho\,\dot{z} - z\,\dot{\rho})\,\sin\phi - m\,\rho\,z\,\dot{\phi}\,\cos\phi \tag{23}$$

$$M_y \;=\; \vec{M} \cdot \hat{y} \;=\; -m\,(\rho\,\dot{z} - z\,\dot{\rho})\,\cos\phi - m\,\rho\,z\,\dot{\phi}\,\sin\phi \tag{24}$$

$$M_z \;=\; \vec{M} \cdot \hat{z} \;=\; m\,\rho^2\,\dot{\phi} \tag{25}$$

Spherical coordinates:

$$M_x \;=\; \vec{M} \cdot \hat{x} \;=\; -m\,r^2\left(\dot{\theta}\,\sin\phi + \dot{\phi}\,\sin\theta\,\cos\theta\,cos\phi\right) \tag{26}$$

$$M_y \;=\; \vec{M} \cdot \hat{y} \;=\; m\,r^2\left(\dot{\theta}\,\cos\phi - \dot{\phi}\,\sin\theta\,\cos\theta\,sin\phi\right) \tag{27}$$

$$M_z \;=\; \vec{M} \cdot \hat{z} \;=\; m\,r^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \tag{28}$$