Solution for assignment 22: Turning points of the spherical pendulum for a special case.

The Energy is

$$E = \frac{m}{2}\,R^2\dot\theta^2 + \frac{m}{2}\,R^2\sin^2(\theta)\dot\phi^2 + U_0\,\cos(\theta) \tag{1}$$

with U_0 given by $U_0 = m\,g\,R = E/2$. The angular momentum

$$M_z = m\,R^2\sin^2(\theta)\,\dot\phi \tag{2}$$

is conserved. Substituting $\dot\phi^2 = M_z^2/(m^2R^4\sin^4\theta)$ the energy becomes

$$E = \frac{m}{2}\,R^2\dot\theta^2 + \frac{M_z^2}{2\,m\,R^2\sin^2\theta} + U_0\,\cos(\theta) \tag{3}$$

with $M_z^2/(2\,m\,R^2) = E$. Turning points are then given by the solutions of

$$0 = \frac{m}{2}\,R^2\dot\theta^2 = E - \frac{E}{\sin^2\theta} - \frac{E}{2}\,\cos\theta \tag{4}$$

$$0 = \sin^2(\theta) - 1 - \frac{1}{2}\,\cos\theta\,\sin^2\theta = \cos^2(\theta) - \frac{1}{2}\,\cos\theta + \frac{1}{2}\,\cos^3\theta\,. \tag{5}$$

With $x = \cos\theta$

$$0 = +x\,\left(x^2 - 2\,x - 1\right) \tag{6}$$

with the solutions $x_0 = 0 \Rightarrow \theta_0 = \pi/2$ and

$$x_{1,2} = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2} \tag{7}$$

of which only $x_2 = 1 - \sqrt{2}$ is in the physical range of $\cos\theta$. So, we find the turning points

$$\theta_{\min} = \cos^{-1}(0) = \frac{\pi}{2} = 1.5707963\ldots\,, \tag{8}$$

$$\theta_{\max} = \cos^{-1}(1-\sqrt{2}) = 1.9978749\ldots\,. \tag{9}$$

As an addtional task a plot of

$$f(\theta) = \frac{E}{\sin^2\theta} + \frac{E}{2}\,\cos\theta - E \tag{10}$$

for, e.g., $E = 1$ is instructive.