ADVANCED DYNAMICS — PHY 4936

Solution 27: (A) Let $\lambda = \omega^2$. We have to solve

$$\det \begin{vmatrix} 5-\lambda & 1-\lambda \\ 1-\lambda & 2(1-\lambda) \end{vmatrix} \;=\; 0\,,$$

which leads to the quadratic equation

$$\begin{aligned} 0 &= (5-\lambda)\,2\,(1-\lambda) - (1-\lambda)\,(1-\lambda) \\ &= 10 - 2\,\lambda - 10\,\lambda + 2\,\lambda^2 - 1 + 2\,\lambda - \lambda^2 \\ &= \lambda^2 - 10\,\lambda + 9 \end{aligned}$$

with the solutions

$$\begin{aligned} \omega_{1,2}^2 &= \lambda_{1,2} \;=\; 5 \pm \sqrt{5^2-9} = 5 \pm 4 \\ \omega_1^2 &= \lambda_1 \;=\; 9\,, \quad \omega_2^2 \;=\; \lambda_2 \;=\; 1\,. \end{aligned}$$

(B) Using normal coordinates $\Theta_1 = \mathrm{Re}\,[C_1\,\exp(i\omega_1 t)]$ and $\Theta_2 = \mathrm{Re}\,[C_2\,\exp(i\omega_2 t)]$, the general solutions are the superpositions

$$\begin{aligned} x_1 &= \triangle_{11}\,\Theta_1 + \triangle_{12}\,\Theta_2 \\ x_2 &= \triangle_{21}\,\Theta_1 + \triangle_{22}\,\Theta_2 \end{aligned}$$

It turns out that we have to take the minor of the second row, because for the first row the $\triangle_{12}$ and $\triangle_{22}$ minors are both zero. Solutions are then

$$\begin{aligned} x_1 &= 8\,\Theta_1 + 0 \;=\; 8\,\Theta_1 \\ x_2 &= -4\,\Theta_1 + 4\,\Theta_2\,. \end{aligned}$$

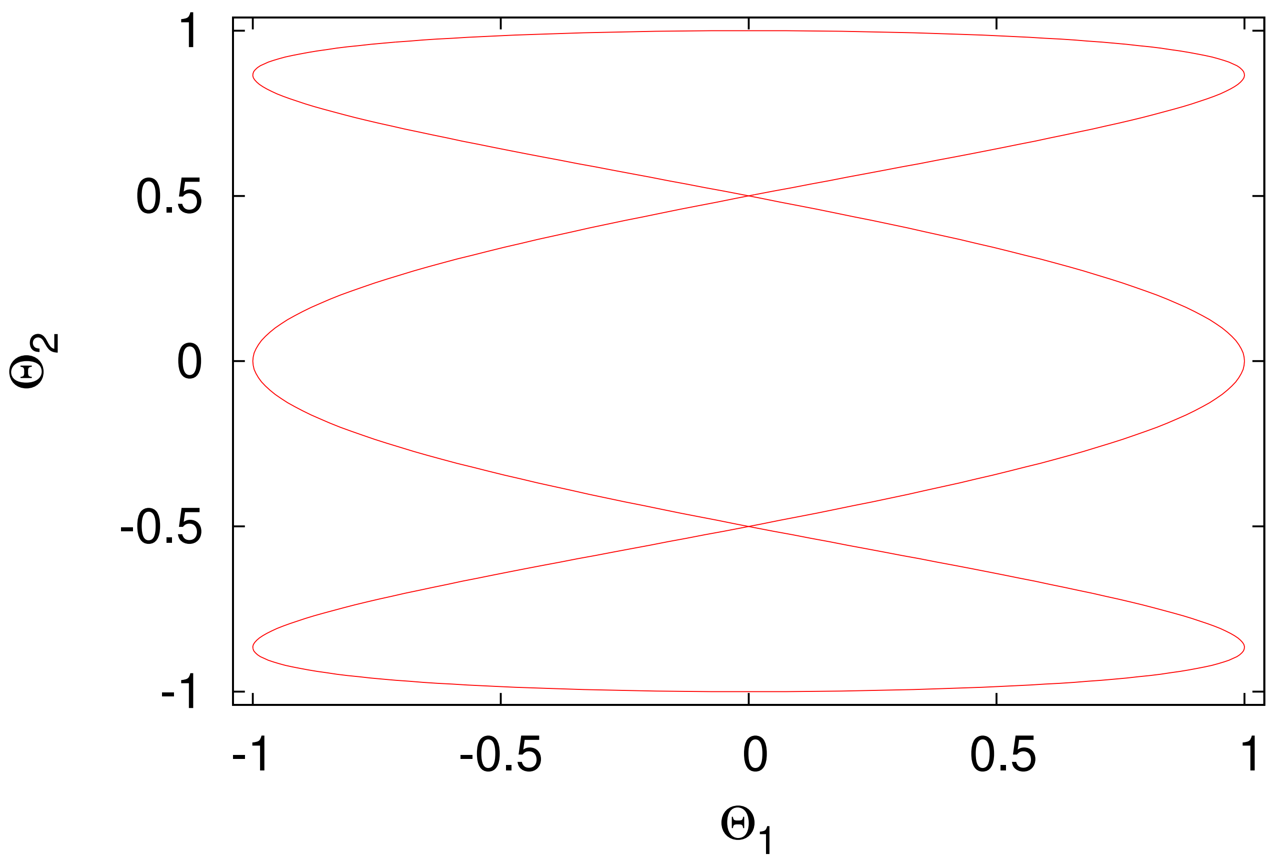
(C) Substituting these equations into the Lagrangian

$$L \;=\; \frac{1}{2}\,(\dot{x}_1\,\dot{x}_1 + \dot{x}_1\,\dot{x}_2 + \dot{x}_2\,\dot{x}_1 + 2\,\dot{x}_2\,\dot{x}_2 - 5\,x_1\,x_1 - x_1\,x_2 - x_2\,x_1 - 2\,x_2\,x_2)$$

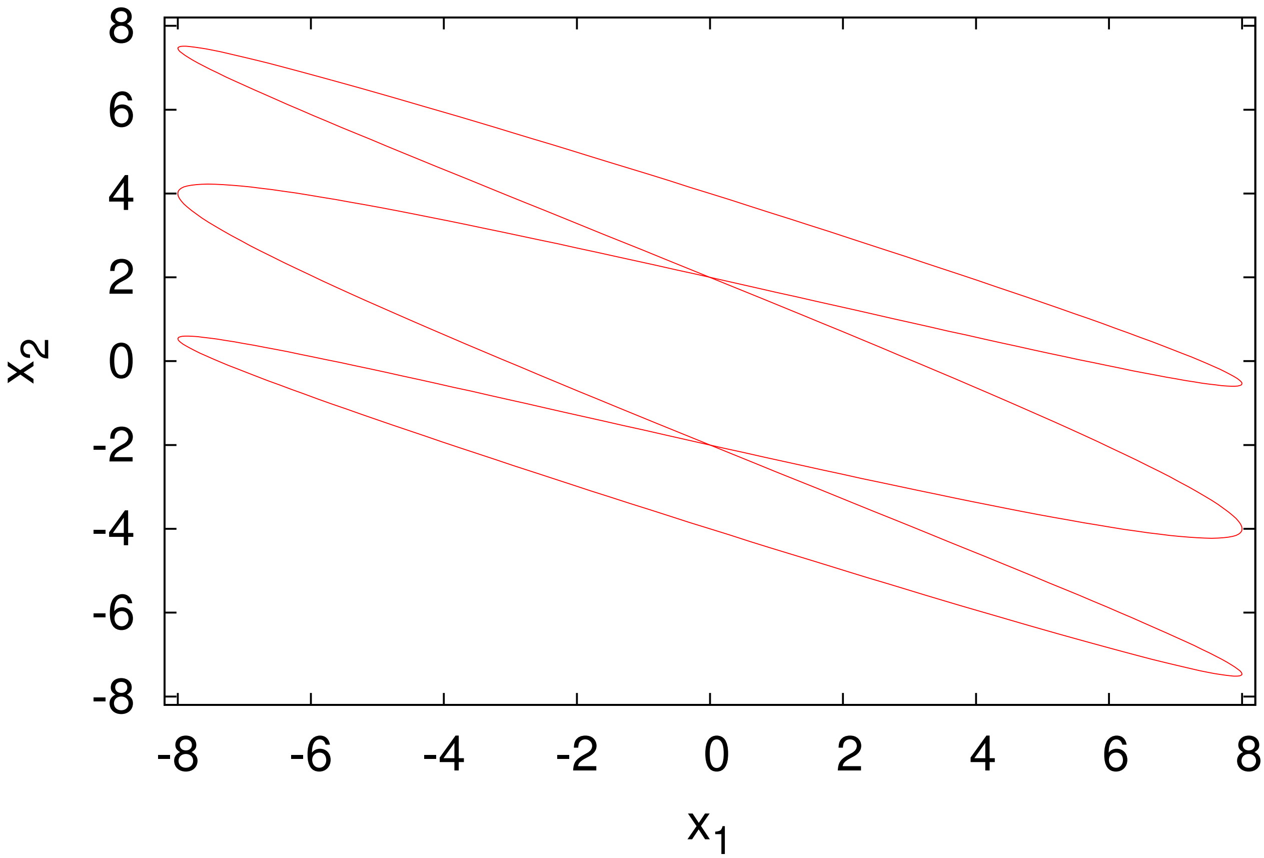
diagonalizes simultaneously both terms and gives

$$L \;=\; 16\,\left(\dot{\Theta}_1\right)^2 + 16\,\left(\dot{\Theta}_2\right)^2 - 144\,(\Theta_1)^2 - 16\,(\Theta_2)^2\;.$$

(D) The initial condition $\Theta_1(0) = 1$ and $\dot{\Theta}_1(0) = 0$ implies $\Theta_1(t) = \cos(\omega_1 t) = \cos(3t)$. and initial condition $\Theta_2(0) = 0$ and $\dot{\Theta}_1(0) = 1$ implies $\Theta_2(t) = \sin(\omega_2 t) = \sin(t)$. Plots in the Θ_1, Θ_2 and x_1, x_2 planes follow.



1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
Θ2
Θ1



8
6
4
2
0
-2
-4
-6
-8
x_2
-8
-6
-4
-2
0
2
4
6
8
x_1