Solution for # 29

Defining ρ as the mass density, we use polar coordinates

$$x = r\cos\theta\,, \qquad y = r\sin\theta\ .$$

Given a coordinate system (x', y') which rotates with the disk (see the figure), the location of the CM is $\bar{x}_{CM} = 0$ and

$$\begin{aligned}
\bar{y}_{CM} &= \frac{\rho}{M}\left\{\int_0^R drr\int_0^\pi d\theta r\sin\theta + 2\int_0^R drr\int_\pi^{2\pi} d\theta r\sin\theta\right\} \\
&= \frac{\rho}{M}\left\{\frac{R^3}{3}2 - \frac{R^3}{3}4\right\} = -\frac{2}{3}\frac{\rho R^3}{M} = -\frac{4R}{9\pi}.
\end{aligned}$$

Where in the last line we have used

$$M = \rho\frac{\pi R^2}{2} + 2\rho\frac{\pi R^2}{2} = \frac{3}{2}\rho\pi R^2.$$

Again referencing the figure, we see that the relationships between the lab coordinates and the coordinates of the center of mass are

$$\left\{\begin{aligned}
x_{CM} &= R\theta - |\bar{y}_{CM}|\sin\theta &&= R\theta - (4R/9\pi)\,\sin\theta \\
y_{CM} &= R - |\bar{y}_{CM}|\cos\theta &&= R - (4R/9\pi)\,\cos\theta.
\end{aligned}\right.$$

$$\Downarrow$$

$$\left\{\begin{aligned}
\dot{x}_{CM} &= R\dot{\theta} - (4R/9\pi)\,\cos\theta\dot{\theta} \\
\dot{y}_{CM} &= (4R/9\pi)\,\sin\theta\dot{\theta}.
\end{aligned}\right.$$

The kinetic energy of the disk is made of 2 terms:

$$T = T_{trans} + T_{rot},$$

where T_{trans} is the translational kinetic energy of the mass M with coordinates (x_{CM}, y_{CM}) and velocity $(\dot{x}_{CM}, \dot{y}_{CM})$, and T_{rot} is the rotational kinetic energy about the center of mass. Thus

$$T = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_3\dot{\theta}^2.$$

Here I_3 is the moment of inertia about the CM, with respect to the z-axis, perpendicular to the plane of the figure. To find I_3 we will calculate it with respect to the center of disk (since it is easier!) and use the parallel axis theorem, (32.12) of Landau and Lifshitz, to obtain it with respect to the CM.

$$\begin{aligned}
I_3\Big|_0 &= \rho\int_0^R drr\int_0^\pi d\theta(x^2+y^2) + 2\rho\int_0^R drr\int_\pi^{2\pi} d\theta(x^2+y^2) \\
&= \rho\frac{R^4}{4}\pi + 2\rho\frac{R^4}{4}\pi = \rho R^4\frac{3}{4}\pi = \frac{1}{2}MR^2 \\
I_3\Big|_{CM} &= I_3|_0 - M\bar{y}_{CM}^2 = \frac{1}{2}MR^2 - M\frac{16}{81}\frac{R^2}{\pi^2} = \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right].
\end{aligned}$$

Then we can calculate the kinetic energy

$$\begin{aligned} T &= T_{trans} + T_{rot} = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_3\Big|_{CM}\dot{\theta}^2 \\ &= \frac{1}{2}MR^2\dot{\theta}^2\left[\left(1 - \frac{4}{9\pi}\cos\theta\right)^2 + \left(\frac{4}{9\pi}\sin\theta\right)^2\right] + \frac{1}{4}MR^2\dot{\theta}^2\left[1 - \frac{32}{81\pi^2}\right] \\ &= \frac{1}{2}MR^2\dot{\theta}^2\left[1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta + \frac{1}{2} - \frac{16}{81\pi^2}\right] \\ &= \frac{1}{2}MR^2\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right]. \end{aligned}$$

The potential energy follows from the center of mass position,

$$U = Mgy_{CM} = MgR\left[1 - \frac{4}{9\pi}\cos\theta\right],$$

and the Lagrangian is the sum of these two terms,

$$L = T - V = \frac{1}{2}MR^2\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right] - MgR\left[1 - \frac{4}{9\pi}\cos\theta\right].$$