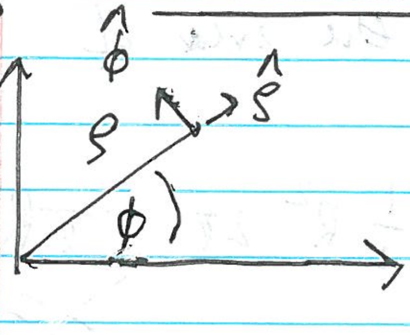
Book p. 98

Cylindrical Coordinates

CC ①



$x = \varrho \cos\phi$

$y = \varrho \sin\phi, \quad z = z$

Local orthonormal unit vectors $\hat\varrho, \hat\phi$ (depend on positions). And fixed $\hat z$. (Global)

Position vector: $\vec r = \varrho\hat\varrho + z\hat z$

General vector: $\vec A = A_\varrho \hat\varrho + A_\phi \hat\phi + A_z \hat z$

with $A_\varrho = \vec A \cdot \hat\varrho$, $A_\phi = \vec A \cdot \hat\phi$, $A_z = \vec A \cdot \hat z$

$d\vec r = \hat\varrho\, d\varrho + \hat\phi\, \varrho\, d\phi + \hat z\, dz$

$(d\vec r)^2 = (ds)^2 = (d\varrho)^2 + \varrho^2 (d\phi)^2 + (dz)^2$

as $\hat\varrho \cdot \hat\phi = 0 = \hat\varrho \cdot \hat z = \hat\phi \cdot \hat z$

$\hat\varrho \cdot \hat\varrho = \hat\phi \cdot \hat\phi = \hat z \cdot \hat z = 1$

Devide!

Nabla Operator: $\vec\nabla = \hat\varrho \frac{\partial}{\partial\varrho} + \frac{\hat\phi}{\varrho}\frac{\partial}{\partial\phi} + \hat z \frac{\partial}{\partial z}$

$\vec V = \frac{d\vec r}{dt} = \hat\varrho\,\dot\varrho + \hat\phi\,\varrho\,\dot\phi + \hat z\,\dot z$ velocity

Area element in x-y plane:

$$d\vec{a} = \hat{\rho}\, d\rho \times \rho\hat{\phi}\, d\phi = \rho\, d\rho\, d\phi\, \hat{z}$$

$$da = d\rho\, (\rho\, d\phi)$$

$$d^3x = d\rho\, (\rho\, d\phi)\, dz$$ Volume element

Complicated (but instructive !?) derivation of the expression for $\vec{v}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\rho\hat{\rho} + z\hat{z}\right)$$

$$= \dot{\rho}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \dot{z}\hat{z} + z\cancel{\frac{d\hat{z}}{dt}}$$

$= 0$ as $\hat{z}$ fixed

Product Rule!

New kid in town!

Elementary geometry:

$$\hat{\rho} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$

$$\begin{array}{c|cc} & \hat{x} & \hat{y} \\ \hat{\rho} & \hat{\rho}\cdot\hat{x} & \hat{\rho}\cdot\hat{y} \\ \hat{\phi} & \hat{\phi}\cdot\hat{x} & \hat{\phi}\cdot\hat{y} \end{array}$$

Backward: $\hat{x}$ ↑ $\hat{y}$ ↑

$$\hat{x} = \cos(\phi)\hat{\rho} - \sin(\phi)\hat{\phi}$$

$$\hat{y} = \sin(\phi)\hat{\rho} + \cos(\phi)\hat{\phi}$$

Now, $\dot{\hat{\rho}} = -\dot{\phi}\sin(\phi)\hat{x} + \dot{\phi}\cos(\phi)\hat{y}$

$$= \cancel{-\dot{\phi}\sin(\phi)\cos(\phi)\hat{\rho}} + \dot{\phi}\sin^2(\phi)\hat{\phi}$$

$$\cancel{+\dot{\phi}\cos(\phi)\sin(\phi)\hat{\rho}} + \dot{\phi}\cos^2(\phi)\hat{\phi} = \dot{\phi}\hat{\phi}$$

→ Example Kepler's 2. Law (→ ④).

Gradient: $\vec{\nabla}\psi = \hat{\rho}\dfrac{\partial \psi}{\partial \rho} + \dfrac{\hat{\phi}}{\rho}\dfrac{\partial \psi}{\partial \phi} + \dfrac{\partial \psi}{\partial z}$

Divergence: $\vec{\nabla}\cdot\vec{A} =$

$$\hat{\rho}\frac{\partial}{\partial \rho}\left(\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z\right)$$

$$+\frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi}\left(\qquad '' \qquad\right)$$

$$+\hat{z}\frac{\partial}{\partial z}\left(\qquad '' \qquad\right)$$

$$= \frac{\partial A_\rho}{\partial \rho} \qquad \left(\text{as } \frac{\partial \hat{\rho}}{\partial \rho} = 0 = \frac{\partial \hat{\phi}}{\partial \rho} = \frac{\partial \hat{z}}{\partial \rho}\right)$$

$= \hat{\rho}\cdot\hat{\phi}$

$= \hat{\rho}\cdot\hat{z}$

$$+ \frac{\partial A_z}{\partial z} \qquad \left(\frac{\partial \hat{\rho}}{\partial z} = 0 \quad \text{etc}\right)$$

$-\hat{\rho} \Rightarrow \hat{\phi}\cdot\hat{\rho} = 0$

$$+\frac{\hat{\phi}}{\rho}\frac{\partial \hat{\rho}}{\partial \phi}A_\rho + \frac{\hat{\phi}}{\rho}\frac{\partial \hat{\phi}}{\partial \phi}A_\phi + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + 0$$

Now careful! Product Rule!

$$\frac{d\hat{\rho}}{d\phi} = -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y} = \hat{\phi}$$

$$\frac{\hat{\phi}\cdot\hat{\phi}}{\rho}A_\rho = \frac{A_\rho}{\rho}$$

$$\frac{d\hat{\phi}}{d\phi} = -\cos(\phi)\,\hat{x} - \sin(\phi)\,\hat{y}$$
$$= -\hat{\rho}$$

Collecting: $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_\rho}{\partial \rho} + \frac{A_\rho}{\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Laplace Operator:

$$\vec{\nabla}^2 \psi = \vec{\nabla}\cdot\vec{\nabla}\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

as vector

Example (p. 104):

Area law of planetary motion from angular momentum conservation.

$\vec{L} = m\,\vec{r}\times\vec{v}$, $\frac{d\vec{L}}{dt} = 0 \Rightarrow$

$\vec{L}$ constant vector. Choose $\vec{L} = L\hat{z}$.

perpendicular $\hat{z}$

$$\vec{L} = m\left(\rho\hat{\rho} + \cancel{z\hat{z}}\right) \times \left(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \cancel{\dot{z}\hat{z}}\right)$$

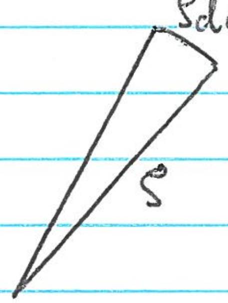
$$= m\rho^2\dot{\phi}\,\underline{\underline{\hat{\rho}\times\hat{\phi}}} = m\rho^2\dot{\phi}\,\underline{\underline{\hat{z}}}$$

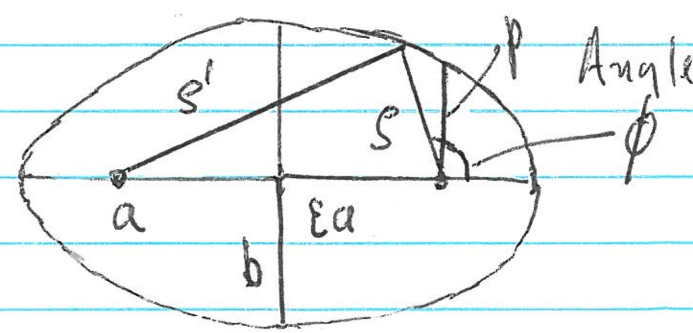
$\uparrow$ $\hat{\rho}\times\hat{\rho} = 0$

$$L_z = m\rho^2\dot{\phi} = \text{constant}$$

$$m\rho^2\, d\phi = dt \times \text{constant}$$

$\frac{1}{2}\rho(\rho\, d\phi)$ area of infinitesimal triangle.





Ellipse:

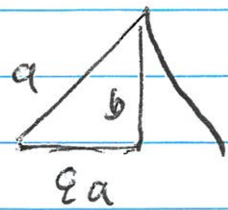
2a major axis

$\rho' + \rho = 2a$ orbit defines ellipse

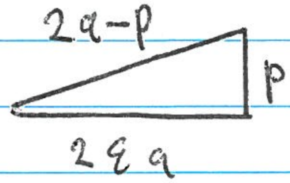
$0 \le \varepsilon < 1$ eccentricity ($\varepsilon = 0$ circle)

$\pm \varepsilon a$ focus points (foci)

p called "lotus rectum"



$$(\varepsilon a)^2 + b^2 = a^2$$
$$b^2 = a^2(1-\varepsilon^2)$$
$$\underline{\underline{b = a\sqrt{1-\varepsilon^2}}}$$



$$(2a-p)^2 = (2\varepsilon a)^2 + p^2$$
$$4a^2 - 4ap + \cancel{p^2} = 4\varepsilon^2 a^2 + \cancel{p^2}$$
$$a - p = \varepsilon^2 a$$
$$\underline{\underline{p = a(1-\varepsilon^2)}}$$

$$\vec{\rho}\,' = \varepsilon a \hat{x} + \vec{\rho}$$

$$(\rho')^2 = (2a\hat{x}-\rho)^2 = 4\varepsilon^2 a^2 + 4\varepsilon a \underbrace{\hat{x}\cdot\vec{\rho}}_{=\rho\cos(\phi)} + \rho^2$$

$$4a^2 - 4a\rho + \cancel{\rho^2} = 4\varepsilon^2 a^2 + 4\varepsilon a\rho\cos\phi + \cancel{\rho^2}$$

$$a - \rho = \varepsilon^2 a + \varepsilon\rho\cos\phi$$

$$\rho = a(1-\varepsilon^2) - \varepsilon\rho\cos\phi$$
$$= p - \varepsilon\rho\cos\phi$$

Ellipse in cylindrical coordinates.

Curl $\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}$, $\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}$

and other derivatives of cylindrical unit vectors with respect to cylindrical coordinates are all zero.

$$\nabla \times \vec{A} = \left(\hat{\rho} \frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(\hat{\rho} A_\rho + \hat{\phi} A_\phi + \hat{z} A_z \right)$$

$$= \hat{\rho} \times \hat{\phi} \frac{\partial A_\phi}{\partial \rho} + \hat{\rho} \times \hat{z} \frac{\partial A_z}{\partial \rho}$$

$$+ \frac{\hat{\phi}}{\rho} \times \hat{\rho} \frac{\partial A_\rho}{\partial \phi} + \frac{\hat{\phi}}{\rho} \times \underline{\underline{\frac{\partial \hat{\phi}}{\partial \phi} A_\phi}} + \frac{\hat{\phi}}{\rho} \times \hat{z} \frac{\partial A_z}{\partial \phi}$$

$$+ \hat{z} \times \hat{\rho} \frac{\partial A_\rho}{\partial z} + \hat{z} \times \hat{\phi} \frac{\partial A_\phi}{\partial z}$$

$$= \hat{z} \frac{\partial A_\phi}{\partial \rho} - \hat{\phi} \frac{\partial A_z}{\partial \rho}$$

$$- \frac{\hat{z}}{\rho} \frac{\partial A_\rho}{\partial \phi} + \underline{\underline{\frac{\hat{z}}{\rho} A_\phi}} + \frac{\hat{\rho}}{\rho} \frac{\partial A_z}{\partial \phi}$$

$$+ \hat{\phi} \frac{\partial A_\rho}{\partial z} - \hat{\rho} \frac{\partial A_\phi}{\partial z}$$

$$= \hat{\rho} \left(\frac{1}{\rho} \frac{\partial}{\partial \phi} A_z - \frac{\partial}{\partial z} A_\phi \right) + \hat{\phi} \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right)$$

$$+ \hat{z} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\rho \right)$$

Use:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{1}{s}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_s & A_\phi & A_z \end{vmatrix} + \frac{\hat{z}}{s} A_\phi$$

instead of the (equivalent) equation in the book. Why!? (p. 111, 2.26)

Metric for curved coordinates: (p. 115)

$$g_{ij} = \frac{\partial x_l}{\partial q_i} \frac{\partial x_l}{\partial q_j}$$

$$(ds)^2 = g_{ij}\, dq_i\, dq_j$$

Skip rest on curved coordinates:

Simpler cases (locally orthonormal) first! //