## Gauss's Theorem

(P.68 book)

$$\oint_S \vec{A}\cdot d\vec{\sigma} = \int_V \vec{\nabla}\cdot\vec{A}\; d^3x$$

As just seen:

$$\sum_{\text{six surfaces}} \vec{A}\cdot d\vec{\sigma} = \vec{\nabla}\cdot\vec{A}\; \underset{\substack{\uparrow \\ d\tau \text{ in book}}}{d^3x}.$$

Fill volume with infinitesimal volume elements:

$$\sum_{\text{interior surfaces}} \vec{A}\cdot d\vec{\sigma} = 0$$

Due to orientations:

$$d\vec{\sigma} = \hat{n}\, dA$$

Therefore,

$$\sum_{\substack{\text{exterior}\\ \text{surfaces}}} \vec{A}\cdot d\vec{\sigma} = \sum_{\substack{\text{infinitesimal}\\ \text{volumes}}} \vec{\nabla}\cdot\vec{A}\; d^3x$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\oint_S \vec{A}\cdot d\vec{\sigma} \qquad\qquad \int_V \vec{\nabla}\cdot\vec{A}\; d^3x$$

Gauss's Theorem ②

**Example: Electric Field**

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Enclosed electric charge

$$\oint_S \vec{E} \cdot d\vec{\sigma} = \frac{1}{\varepsilon_0} \int_V \rho \, d\tau = \frac{Q}{\varepsilon_0}$$

---

p.70 Book

**Green's Theorem:** ($u, v$ scalar function)

$$\vec{\nabla} (u \vec{\nabla} v) = u \vec{\nabla}^2 v + (\vec{\nabla} u)(\vec{\nabla} v) \quad (1)$$

$$\vec{\nabla} (v \vec{\nabla} u) = v \vec{\nabla}^2 u + (\vec{\nabla} v)(\vec{\nabla} u)$$

$$\vec{\nabla} \left[ u \vec{\nabla} v - v \vec{\nabla} u \right] = u \vec{\nabla}^2 v - v \vec{\nabla}^2 u$$

$$\int_V d^3x \, ( u ) = \int_V d^3x \left( u \vec{\nabla}^2 v - v \vec{\nabla}^2 u \right)$$

$$\parallel$$

$$\oint_S \left[ u \vec{\nabla} v - v \vec{\nabla} u \right] \cdot d\vec{\sigma}$$

$$= \oint_S \left[ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right] dA$$

$$\vec{\nabla} v \cdot d\vec{\sigma} = (\vec{n} \cdot \vec{\nabla} v) \cdot dA = \frac{\partial v}{\partial n} \cdot dA$$

Green's first identity: From (1)

$$\oint_S (u \vec{\nabla} v) \cdot d\vec{r} = \int_V d^3x \left[ u \nabla^2 v + (\vec{\nabla} u)(\vec{\nabla} v) \right]$$

$$\parallel$$

$$\oint_S u \frac{\partial v}{\partial n} dA$$

[Added (not in book, see Jackson)]

## Dirichlet and Neumann Boundary Condition

We show the uniqueness of the solution of the Poisson equation

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$$

with Dirichlet or Neumann BC.

Assume $\Phi_1$ and $\Phi_2$ are both solutions.

Then, $\nabla^2 U = 0$ with $U = \Phi_2 - \Phi_1$

From Green's first identity ($u = v = U$)

$$\oint_S U \frac{\partial U}{\partial n} dA = \int_V (\underbrace{\cancel{U \nabla^2 U}}_{=0} + \vec{\nabla} U \cdot \vec{\nabla} U) \, d^3x$$

$$= \int_V |\vec{\nabla} U|^2 \, d^3x$$

Dirichlet BC: $\Phi_2 = \Phi_1$ on $S$.

Neumann BC: $\dfrac{\partial \Phi_2}{\partial n} = \dfrac{\partial \Phi_1}{\partial n}$ on $S$.

With both BC: $\int_V |\vec{\nabla} U|^2 d^3x = 0$

$\Rightarrow \quad \vec{\nabla} u = 0$

$\Rightarrow \quad u = \text{const}$ with $V$

Dirichlet BC: $u = 0$, solution unique.

Neumann BC: Solution unique up to additive constant.

Homework Example: Paul Trap.

Stoke's Theorem: (Book p.72/3)

$$\oint_C \vec{A} \cdot d\vec{s} = \int_S \vec{\nabla} \times \vec{A} \cdot d\vec{\sigma}$$

$C$: Closed Contour

$S$: An surface with C as boundary

Proof:

For an infinitesimal square: 

$$\sum_{\text{four sides}} \vec{A} \cdot d\vec{s} = \vec{\nabla} \times \vec{A} \cdot d\vec{\sigma}$$

four sides shown before.

See figures (book) 1.26 and 1.35.

Interior paths cancel out.

$$\sum_{\text{exterior lines}} \vec{A} \cdot d\vec{s} = \sum_{\text{rectangels}} \vec{\nabla} \times \vec{A} \cdot d\vec{\sigma}$$

$\downarrow$

$$\oint_C \vec{A} \cdot d\vec{s} = \int_S \vec{\nabla} \times \vec{A} \cdot d\vec{\sigma}$$

**Example:** Conservative Force

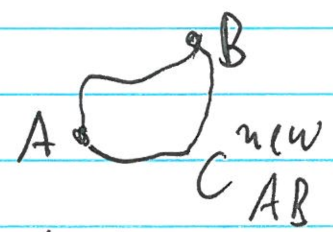
$$\vec{\nabla} \times \vec{F} = 0 \iff \int_C \vec{F} \cdot d\vec{s} = 0$$

We have already seen

$$\vec{F} = -\nabla \phi \quad \text{(Potential)} \Rightarrow \vec{\nabla} \times \vec{F} = 0.$$

The other way round

$$\text{Work by force} = \int_{C_{AB}} \vec{F} \cdot d\vec{r} = \phi(A) - \phi(B)$$




defines now uniquely a potential difference as $\int_{C_{AB}} \vec{F} \cdot d\vec{r} = \int_{C^{new}_{AB}} \vec{F} \cdot d\vec{r}$

as $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Choose $\phi(B)$ and the potential is unique. E.g. $\phi(\infty) = 0$ (Kepler Probl.)

**Example:** Integration of Maxwell's eqn. $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$$\int_{C=\partial S} \vec{E} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\sigma = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{\sigma}$$

↑

symbol for boundary of $S$

$$\frac{d\vec{B}}{dt} = \sum_{i=1}^{3} \frac{\partial \vec{B}}{\partial x_i} \dot{x}_i + \frac{\partial \vec{B}}{\partial t}$$

$$\parallel$$

$(\vec{v} \cdot \vec{\nabla}) \vec{B} \approx 0$ in limit $|\vec{v}| << c$.

In non-relativistic limit $\dfrac{d\vec{B}}{dt} = \dfrac{\partial \vec{B}}{\partial t}$

$$\int_C \vec{E} \cdot d\vec{s} = -\frac{d\Phi}{dt}, \quad \Phi = \int_S \vec{B} \cdot d\vec{\sigma}$$

Faraday's Law.