Dirac δ-distribution δ(x):

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$

and for every continuous test function $f(x)$

$$I = \int_{-\infty}^{+\infty} f(x)\,\delta(x) = f(0)$$

holds, where I is defined by the integrations of a sequence of square integrable functions $\delta_n(x)$,

$$\int_{-\infty}^{+\infty} |\delta_n(x)|^2 \, dx = \text{finite}$$

as

$$I = \lim_{n\to\infty} \underbrace{\int_{-\infty}^{+\infty} f(x)\,\delta_n(x)}_{I_n \text{ (a number)}} = f(0)$$

Example of definitions of $\delta_n(x)$ are given in Fig. 1.44–1.47 of the book, p.87/88.

E.g., box:

$$\delta_n(x) = \begin{cases} n & \text{for } |x| \le \frac{1}{2n}, \\ 0 & \text{for } |x| > \frac{1}{2n}. \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta_n(x) f(x)\, dx = n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(\bar{x}_n)\, dx$$

where $-\frac{1}{2n} < \bar{x} < \frac{1}{2n}$. Therefore

$$I = \lim_{n \to \infty} \left[n\, f(\bar{x}_n) \left(\frac{1}{2n} + \frac{1}{2n} \right) \right]$$

$$= \lim_{n \to \infty} f(\bar{x}_n) = \underline{\underline{f(0)}} \quad \#$$

Charge density $\rho(\vec{r})$ of potential. (Book p.90)

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{e_j}{|\vec{r} - \vec{r}_j|} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}\, d^3x'$$

We have already shown: $\vec{\nabla}^2 \phi(\vec{r}) = 0$ for $\vec{r} \neq \vec{r}_j$.

Now, by Gauss's Law

$$\varepsilon_0^{-1} Q = \varepsilon_0^{-1} \sum_j e_j = \oint_{\text{Surface}} \vec{E} \cdot d\vec{\sigma}$$

$$= -\oint_S \vec{\nabla} \phi \cdot d\vec{\sigma} = -\int \vec{\nabla}^2 \phi \, d\tau$$

Gauss's theorem

$$= \varepsilon_0^{-1} \int \rho(\vec{r}) \, d^3x = \varepsilon_0^{-1} \sum_j e_j$$

Poisson eqn.

Therefore, $\rho(\vec{r}) = \sum_j e_j \, \delta(\vec{r} - \vec{r}_j)$

$$-\vec{\nabla}^2 \frac{1}{|\vec{r} - \vec{r}_j|} = 4\pi \, \delta(\vec{r} - \vec{r}_j)$$

with $\delta(\vec{r} - \vec{r}_j) = \delta(x - x_j)\, \delta(y - y_j)\, \delta(z - z_j)$

An Important Formula:

$$\delta(g(x)) = \sum_a \frac{\delta(x - x_a)}{|g'(x_a)|}$$

$g(x)$ is a continuously differentiable function with simple zeros, i.e., $g'(a) \neq 0$, at $x = x_a$, $a = 1, \ldots, n$.

Proof: 1. $\delta(cx) = \dfrac{\delta(x)}{|c|}$

Let $\varepsilon > 0$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(cx)\,dx = \int_{-\varepsilon}^{+\varepsilon} f(x)\,\delta(cx)\,dx$$

$$= \int_{-\varepsilon c}^{+\varepsilon c} f\left(\frac{x'}{c}\right)\delta(x')\,\frac{dx'}{c} = \int_{-\varepsilon|c|}^{+\varepsilon|c|} f\left(\frac{x'}{c}\right)\frac{\delta(x')}{|c|}\,dx'$$

$x = \dfrac{x'}{c}$

$$= \frac{f(0)}{|c|} = \int_{-\infty}^{+\infty} f(x)\,\frac{\delta(x)}{|c|}\,dx$$

2. Therefore, $\delta\left[c\,(x - x_a)\right] = \dfrac{\delta(x - x_a)}{|c|}$

3. $$\int_{-\infty}^{+\infty} f(x)\,\delta[g(x)]\,dx =$$

$$\sum_a \int_{x_a-\varepsilon}^{x_a+\varepsilon} f(x)\,\delta[g(x)]\,dx =$$

$$\sum_a \int_{x_a-\varepsilon}^{x_a+\varepsilon} f(x)\,\delta[g'(x_a)(x-x_a)]\,dx =$$

$$\sum_a \int_{x_a-\varepsilon}^{x_a+\varepsilon} f(x)\,\frac{\delta(x-x_a)}{|g'(x_a)|}\,dx =$$

$$= \int_{-\infty}^{+\infty} f(x) \sum_a \frac{\delta(x-x_a)}{|g'(x_a)|}\,dx$$

$\square$

Example 1.14.3 (p. 91):

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x^2-2)\,dx = \int_{-\infty}^{+\infty} f(x)\,\delta\left[(x-\sqrt{2})(x+\sqrt{2})\right]dx$$

$$= \int_{-\infty}^{+\infty} f(x)\left[\frac{\delta(x-\sqrt{2})}{2\sqrt{2}} + \frac{\delta(x+\sqrt{2})}{2\sqrt{2}}\right]dx$$

$$= \frac{f(\sqrt{2})}{2\sqrt{2}} + \frac{f(-\sqrt{2})}{2\sqrt{2}}$$

Example 1.14.4 (p.91): Phase Space

$$\int d^4p\, \delta(p_0^2 - \vec{p}^{\,2} - m^2)\, f(p) =$$

$$\int d^3p \int_{-\infty}^{+\infty} dp_0\, \delta(p_0^2 - \vec{p}^{\,2} - m^2)\, f(p_0, \vec{p}) =$$

$$\int d^3p \int_{-\infty}^{+\infty} dp_0\, \delta\left[\left(p_0 - \sqrt{\vec{p}^{\,2} + m^2}\right)\left(p_0 + \sqrt{\vec{p}^{\,2} + m^2}\right)\right] f(p_0, \vec{p}) =$$

$$\int d^3p \int_{-\infty}^{+\infty} dp_0 \left[\frac{\delta\left(p_0 - \sqrt{\vec{p}^{\,2} + m^2}\right)}{2\sqrt{\vec{p}^{\,2} + m^2}} + \frac{\delta\left(p_0 + \sqrt{\vec{p}^{\,2} + m^2}\right)}{2\sqrt{\vec{p}^{\,2} + m^2}}\right]$$

$$\times f(p_0, \vec{p})$$

$$= \int d^3p \left[\frac{f\left(\sqrt{\vec{p}^{\,2} + m^2}, \vec{p}\right)}{2\sqrt{\vec{p}^{\,2} + m^2}} + \frac{f\left(-\sqrt{\vec{p}^{\,2} + m^2}, \vec{p}\right)}{2\sqrt{\vec{p}^{\,2} + m^2}}\right] \square$$

Derivative of the δ-function:

$$\int_{-\infty}^{+\infty} f(x)\,\delta'(x-x')\,dx = -f'(x')$$

|| Integration by parts

$$\underbrace{\cancel{f(x)\,\delta(x-x')\Big|_{x=-\infty}^{x=-\infty}}}_{0} - \int_{-\infty}^{+\infty} f'(x)\,\delta(x-x')\,dx$$

$$= -f'(x)$$

$$\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) \quad \text{Cartesian Coordinates}$$

$$= \frac{\delta(r)}{r^2}\,\delta(\cos\theta)\,\delta(\phi) \quad \text{Spherical Coordinates.}$$

$$\int_0^{\infty} r^2\,dr \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi$$