Mathematical Physics — PHZ 3113

Curl Classwork (February 1, 2013)

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1. Use the Levi-Civita tensor to calculate $\nabla \times \vec{r}$, where $\vec{r}$ is the position vector.

   Solution (with Einstein convention):

$$\begin{aligned} \nabla \times \vec{r} &= \epsilon_{ijk} \hat{x}_i \partial_j x_k \;=\; \epsilon_{ijk} \hat{x}_i \delta_{jk} \\ &= \epsilon_{ikk} \hat{x}_i \;=\; 0 \;. \end{aligned}$$

2. Calculate $\nabla \times \vec{r}\, f(r)$, where $\vec{r}$ is the position vector and $r = |\vec{r}|$.

   Solution:

$$\begin{aligned} \nabla \times \vec{r}\, f(r) &= f(r)\, \nabla \times \vec{r} + [\nabla f(r)] \times \vec{r} \\ &= \frac{df(r)}{d\, r} \hat{r} \times \vec{r} \;=\; 0 \;. \end{aligned}$$

3. Use the Levi-Civita tensor to calculate $\nabla \times \nabla f$, where f is an arbitrary scalar function.

   Solution (with Einstein convention):

$$\nabla \times \nabla f \;=\; \epsilon_{ijk} \hat{x}_i \partial_j \partial_k\, f \;=\; 0$$

because ϵ_{ijk} is anti-symmetric and $\partial_j\partial_k$ is symmetric in the indices j and k.

4. Derive the wave equation for the magnetic field $\vec{B}$ from Maxwell's equations in vacuum

$$\nabla \cdot \vec{B} = 0\,, \quad \nabla \cdot \vec{E} = 0\,,$$
$$\nabla \times \vec{B} = \epsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t}\,, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,.$$

Solution:

$$\begin{aligned} \nabla \times \left(\nabla \times \vec{B}\right) &= \epsilon_0\mu_0 \frac{\partial}{\partial t} \nabla \times \vec{E} \\ \nabla \left(\nabla \cdot \vec{B}\right) - \nabla^2 \vec{B} &= -\epsilon_0\mu_0 \left(\frac{\partial}{\partial t}\right)^2 \vec{B} \\ \nabla^2 \vec{B} &= \frac{1}{c^2} \left(\frac{\partial}{\partial t}\right)^2 \vec{B}\,. \end{aligned}$$

5. Calculate $\nabla \times \vec{F}$ for $\vec{F} = -\hat{x}_1 x_2 + \hat{x}_2 x_1$.

Solution (with Einstein convention):

$$\begin{aligned}
\nabla \times \vec{F} &= -\epsilon_{ij1}\hat{x}_i \partial_j x_2 + \epsilon_{ij2}\hat{x}_i \partial_j x_1 \\
&= -\epsilon_{ij1}\hat{x}_i \delta_{j2} + \epsilon_{ij2}\hat{x}_i \delta_{j1} \\
&= -\epsilon_{i21}\hat{x}_i + \epsilon_{i12}\hat{x}_i \\
&= -\epsilon_{321}\hat{x}_3 + \epsilon_{312}\hat{x}_3 \; = \; 2\,\hat{x}_3 \, .
\end{aligned}$$