Mathematical Physics — PHZ 3113

Einstein Convention Homework 1

1. Use the $3D$ identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad (1)$$

to calculate

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{b}) \qquad (2)$$

where $\hat{a}$ and $\hat{b}$ are unit vectors. Eliminate all $\hat{a} \cdot \hat{b}$ in favor of $\cos\theta$.

Solution: Let

$$\hat{a} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} \quad \text{and} \quad \hat{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{pmatrix}. \qquad (3)$$

Then

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{b}) = \epsilon_{ijk}\epsilon_{ilm}\tilde{a}_j\tilde{b}_k\tilde{a}_l\tilde{b}_m$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\,\tilde{a}_j\tilde{b}_k\tilde{a}_l\tilde{b}_m$$

$$= \left(\tilde{a}_l\tilde{b}_m\tilde{a}_l\tilde{b}_m - \tilde{a}_m\tilde{b}_l\tilde{a}_l\tilde{b}_m\right)$$

$$= (\hat{a}\cdot\hat{a})\left(\hat{b}\cdot\hat{b}\right) - \left(\hat{a}\cdot\hat{b}\right)\left(\hat{b}\cdot\hat{a}\right)$$
$$= 1 - \cos^2(\theta) \;=\; \sin^2(\theta)\,. \qquad (4)$$

Einstein Convention Homework 2

1. Use the identity (1) to eliminate the vector products from the expression

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) \qquad (5)$$

Solution (compare book p.33):

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)$$
$$= \epsilon_{ijk}\hat{x}_i a_j \epsilon_{klm} b_l c_m$$
$$= \epsilon_{kij}\epsilon_{klm}\hat{x}_i a_j b_l c_m$$
$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\hat{x}_i a_j b_l c_m$$
$$= \hat{x}_i a_j b_i c_j - \hat{x}_i a_j b_j c_i$$
$$= \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\left(\vec{a}\cdot\vec{b}\right)\,.$$

2. Use the definition

$$\left(\vec{b}\times\vec{c}\right)_i \;=\; \epsilon_{ijk} b_j c_k \qquad (6)$$

of the i^{th} component of the vector product $\vec{b} \times \vec{c}$ to proof

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right) \ . \quad (7)$$

Solution (compare book p.30):

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \epsilon_{ijk} a_i b_j c_k$$

$$= \epsilon_{jki} b_j c_k a_i = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= \epsilon_{kij} c_k a_i b_j = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right) \ .$$