Mathematical Physics — PHZ 3113
Classwork 12 (April 3, 2013)
Solution Jacobi Determinant

1. Calculate the Jacobi determinant for the transformation from Cartesian to cylindrical coordinates.

Solution:

$$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = z$$

yields the Jacobi determinant

$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho\,.$$

2. Calculate the Jacobi determinant for the transformation from Cartesian to spherical coordinates.

Solution:

$$\begin{aligned} x &= r\sin\theta\cos\phi\,, \\ y &= r\sin\theta\sin\phi\,, \\ z &= r\cos\theta \end{aligned}$$

yields the Jacobi determinant

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= r^2\Big(\sin^3\theta\sin^2\phi + \cos^2\theta\sin\theta\cos^2\phi$$

$$+ \sin^3\theta\cos^2\phi + \cos^2\theta\sin\theta\sin^2\phi\Big)$$

$$= r^2\Big(\sin^3\theta + \cos^2\theta\sin\theta\Big) \;=\; r^2\sin\theta\,.$$