Mathematical Physics — PHZ 3113

Levi-Cevita Tensor 2 Applications

(January 14, 2013)

Group #

Participating students (print):

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1. Write down the values of the cyclic permutations of ϵ_{123} and then of ϵ_{213}. Do you get all $3D$ values this way? Which are positive and which are negative?

$$\epsilon_{123} = +1\,,\ \epsilon_{231} = +1\,,\ \epsilon_{312} = +1\,, \quad (1)$$

$$\epsilon_{132} = -1\,,\ \epsilon_{321} = -1\,,\ \epsilon_{213} = -1\,. \quad (2)$$

We get all 6 non-zero values. The cyclic permutations of ϵ_{123} are $+1$ and the cyclic permutations of ϵ_{213} are -1.

2. In $3D$

$$\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{1jk}\, a_j\, b_k \,=\, a_2 b_3 - a_3 b_2\,, \quad (3)$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{2jk}\, a_j\, b_k = a_3 b_1 - a_1 b_3 \,, \tag{4}$$

$$\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{3jk}\, a_j\, b_k = a_1 b_2 - a_2 b_1 \,. \tag{5}$$

3. In $3D$

$$\begin{aligned} \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\, \hat{x}_i\; a_j\, b_k &= \\ \hat{x}_1\,(a_2 b_3 - a_3 b_2) &+ \\ \hat{x}_2\,(a_3 b_1 - a_1 b_3) &+ \\ \hat{x}_3\,(a_1 b_2 - a_2 b_1) &= \\ \vec{a} \times \vec{b}\,. \end{aligned} \tag{6}$$

Definition of the determinant of a nD matrix:

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ . & & . \\ . & & . \\ . & & . \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \tag{7}$$

$$\sum_{i_1=1}^{n} \dots \sum_{i_n=1}^{n} \epsilon_{i_1 \dots i_n}\, a_{1i_1} \dots a_{ni_n} \,.$$

4. In $2D$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\epsilon_{ij}\,a_{1i}a_{2j} = \epsilon_{12}a_{11}a_{22} + \epsilon_{21}a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}\,. \tag{8}$$

5. In $3D$ (using cyclic permutations)

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,a_{1i}a_{2j}a_{3k} = +a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \tag{9}$$

Notice that this corresponds to a well known rule for evaluating the determinant of a 3×3 matrix.

6. Substitute in the previous expression

$$a_{11} = \hat{x}_1,\ a_{12} = \hat{x}_2,\ a_{13} = \hat{x}_3\,. \tag{10}$$

It becomes

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,\hat{x}_i a_{2j}a_{3k} = \hat{x}_1\left(a_{22}a_{33} - a_{23}a_{32}\right) + \hat{x}_2\left(a_{23}a_{31} - a_{21}a_{33}\right) + \hat{x}_3\left(a_{21}a_{32} - a_{22}a_{31}\right)\,. \tag{11}$$

7. Substitute in the previous expression

$$a_{21} = a_1\,,\; a_{22} = a_2\,,\; a_{23} = a_3\,, \quad (12)$$

$$a_{31} = b_1\,,\; a_{32} = b_2\,,\; a_{33} = b_3\,. \quad (13)$$

It becomes

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\,\hat{x}_i a_j a_k = \quad (14)$$

$$\hat{x}_1\,(a_2b_3 - a_3b_2) +$$

$$\hat{x}_2\,(a_3b_1 - a_1b_3) +$$

$$\hat{x}_3\,(a_1b_2 - a_2b_1) \;\; .$$

8. In $3D$, write $\vec{a} \times \vec{b}$ as determinant (book p.23).

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (15)$$

9. **Proof the $3D$ identity**

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad (16)$$

by calculating the expression for the nine possibilites of values for jk, i.e., 11, 12, 13, 21, 22, 23, 31, 32, 33.

$$\sum_{i=1}^{3} \epsilon_{i11}\epsilon_{ilm} \;=\; 0 = \delta_{1l}\delta_{1m} - \delta_{1m}\delta_{1l}\,,$$
$$\sum_{i=1}^{3} \epsilon_{i12}\epsilon_{ilm} \;=\; \epsilon_{312}\epsilon_{3lm} = \delta_{1l}\delta_{2m} - \delta_{1m}\delta_{2l}\,.$$

Now there are only two non-zero cases: $lm = 12$ and $lm = 21$. One sees immediately that left and right side of the last equal sign agree in each case. Similarly, this holds for the other values discussed below.

$$\sum_{i=1}^{3} \epsilon_{i13}\epsilon_{ilm} \;=\; \epsilon_{213}\epsilon_{2lm} = \delta_{1l}\delta_{3m} - \delta_{1m}\delta_{3l}\,,$$
$$\sum_{i=1}^{3} \epsilon_{i21}\epsilon_{ilm} \;=\; \epsilon_{321}\epsilon_{3lm} = \delta_{2l}\delta_{1m} - \delta_{2m}\delta_{1l}\,,$$
$$\sum_{i=1}^{3} \epsilon_{i22}\epsilon_{ilm} \;=\; 0 = \delta_{2l}\delta_{2m} - \delta_{2m}\delta_{2l}\,,$$
$$\sum_{i=1}^{3} \epsilon_{i23}\epsilon_{ilm} \;=\; \epsilon_{123}\epsilon_{1lm} = \delta_{2l}\delta_{3m} - \delta_{2m}\delta_{3l}\,,$$
$$\sum_{i=1}^{3} \epsilon_{i31}\epsilon_{ilm} \;=\; \epsilon_{231}\epsilon_{2lm} = \delta_{3l}\delta_{1m} - \delta_{3m}\delta_{1l}\,,$$
$$\sum_{i=1}^{3} \epsilon_{i32}\epsilon_{ilm} \;=\; \epsilon_{132}\epsilon_{1lm} = \delta_{3l}\delta_{2m} - \delta_{3m}\delta_{2l}\,,$$

$$\sum_{i=1}^{3} \epsilon_{i33}\epsilon_{ilm} = 0 = \delta_{3l}\delta_{3m} - \delta_{3m}\delta_{3l} \, .$$