Mathematical Physics — PHZ 3113

Vector Integration Homework

(February 20, 2013)

Integrate (explicitly!) the force

$$\vec{F} = -x_2\,\hat{x}_1 + x_1\,\hat{x}_2 \tag{1}$$

in anti-clockwise direction along the boundary of an equilateral triangle in the $x_1 - x_2$ plane. The triangle has a side from $(0,0)$ to $(a,0)$ on the x_1 axis and closes in the upper half of the $x_1 - x_2$ plane.

Solution: The sides of the triangle are given by the vectors

$$\begin{aligned} \vec{s}_1 &= \vec{0} \;=\; (0,0)\,, \\ \vec{s}_2 &= a\,\hat{x}_1\,, \\ \vec{s}_3 &= \frac{1}{2}\,a\,\hat{x}_1 + \frac{\sqrt{3}}{2}\,a\,\hat{x}_2\,. \end{aligned}$$

Therefore,

$$\oint_{\triangle} d\vec{s}\cdot\vec{F} \;=$$

$$\int_{\vec{s}_1}^{\vec{s}_2} d\vec{s}\cdot\vec{F} + \int_{\vec{s}_2}^{\vec{s}_3} d\vec{s}\cdot\vec{F} + \int_{\vec{s}_3}^{\vec{s}_1} d\vec{s}\cdot\vec{F}$$

where

$$\int_{\vec{s}_1}^{\vec{s}_2} d\vec{s}\cdot\vec{F} \;=\; \int_0^a dx_1\, 0 \;=\; 0\,, \qquad (2)$$

$$\int_{\vec{s}_2}^{\vec{s}_3} d\vec{s}\cdot\vec{F} \;=\; \int_0^{a\sqrt{3}/2} dx_2\, x_1(x_2)$$

$$-\;\int_a^{a/2} dx_1\, x_2(x_1)$$

with

$$x_1(x_2) \;=\; a - \frac{x_2}{\sqrt{3}}\,, \quad x_2(x_1) \;=\; (a - x_1)\sqrt{3}\,.$$

Hence,

$$\int_0^{a\sqrt{3}/2} dx_2\, x_1(x_2) \;=\; \int_0^{a\sqrt{3}/2} dx_2\, \left(a - \frac{x_2}{\sqrt{3}}\right)$$

$$=\; \frac{\sqrt{3}}{2}\, a^2 - \frac{\sqrt{3}}{8}\, a^2 \;=\; \frac{3\sqrt{3}}{8}\, a^2$$

$$-\int_a^{a/2} dx_1\, x_2(x_1) \;=\; \sqrt{3}\int_a^{a/2} dx_1\, (x_1 - a)$$

$$= \; a^2 \left(\frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \sqrt{3} \right) = \frac{\sqrt{3}}{8}\, a^2 \,,$$

$$\int_{\vec{s}_1}^{\vec{s}_2} d\vec{s} \cdot \vec{F} \; = \; \frac{\sqrt{3}}{2}\, a^2 \,. \qquad (3)$$

Next, along the same lines

$$\int_{\vec{s}_3}^{\vec{s}_1} d\vec{s} \cdot \vec{F} \; = \; \int_{a\sqrt{3}/2}^{0} dx_2\, x_1(x_2) \; - \; \int_{a/2}^{0} dx_1\, x_2(x_1)$$

with

$$x_1(x_2) \; = \; \frac{x_2}{\sqrt{3}} \,, \quad x_2(x_1) \; = \; x_1 \sqrt{3} \,.$$

Hence,

$$\int_{a\sqrt{3}/2}^{0} dx_2\, x_1(x_2) \; = \; \int_{a\sqrt{3}/2}^{0} dx_2\, \frac{x_2}{\sqrt{3}} = -\frac{\sqrt{3}}{8}\, a^2 \,,$$

$$- \int_{a/2}^{0} dx_1\, x_2(x_1) \; = \; \sqrt{3} \int_{0}^{a/2} dx_1\, x_1 = \frac{\sqrt{3}}{8}\, a^2 \,,$$

$$\int_{\vec{s}_2}^{\vec{s}_3} d\vec{s} \cdot \vec{F} \, = \, 0 \, . \tag{4}$$

Collecting the integrals (2), (3) and (4) we arrive at the final result

$$\oint_{\triangle} d\vec{s} \cdot \vec{F} = \frac{\sqrt{3}}{2} \, a^2 \, .$$