Practical Next-to-Leading Order calculation

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Outline of this lecture

- Setting the frame: basic concepts and terminology.
- Structure of a Next-to-Leading (NLO) calculation:
  - ⟶ Virtual and real corrections, how to approach them: methods, challenges, new developments.
  - ⟶ Cancelling UV and IR divergencies.
  - ⟶ Convoluting with initial state Parton Distribution Functions.
  - ⟶ First result: naive NLO parton-level Monte Carlo.
- Examples of NLO results: what do we gain?
- When is NLO not enough? Examples of physical observables that require various levels of improvement:
  - ⟶ one more order in the perturbative expansion: going Next-to-Next-to-Leading Order (NNLO);
  - ⟶ resummation of large corrections at all orders: reordering the perturbative expansion.

Setting the Frame

- Hadron colliders (Tevatron, LHC) are the present and close future of particle physics: emphasis on QCD. (see W.-K. Tung's lectures)

- We will learn about the properties of NLO calculations by considering:

  $\longrightarrow$ prototype process: QCD top quark pair production, $q\bar{q}, gg \to t\bar{t}$.
  (see C. Oleari's lectures on Heavy Quark Production)

  $\longrightarrow$ first order of QCD corrections;

  $\longrightarrow$ total/differential cross-sections.

- I will assume a basic knowledge of some fundamental topics of Quantum Field Theory as encountered in Quantum Electrodynamics:

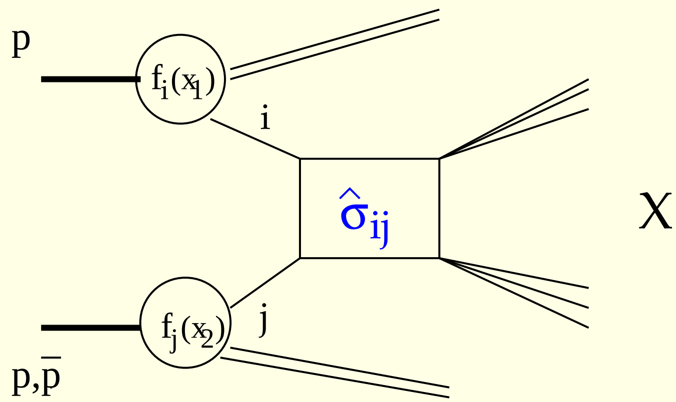
  $\longrightarrow$ perturbative calculation of cross-section from Feynman diagrams;

  $\longrightarrow$ origin of ultraviolet (UV) and infrared (IR) divergences;

  $\longrightarrow$ regularization and renormalization of UV divergences;

  $\longrightarrow$ cancellation of IR divergences for IR-safe observables.

The basic picture of a $p\bar{p}, pp \to X$ high energy process is ...



where the short and long distance part of the QCD interactions can be factorized and the cross section for $pp, p\bar{p} \to X$ can be calculated as:

$$d\sigma(pp, p\bar{p} \to X) = \sum_{ij} \int dx_1 dx_2 f_i{}^p(x_1, \mu_F) f_j{}^{p,\bar{p}}(x_2, \mu_F) d\hat{\sigma}(ij \to X, x_1, x_2, Q^2, \mu_F)$$

$\longrightarrow$ $ij \to$ quarks or gluons (partons)

$\longrightarrow$ $f_i{}^p(x)$, $f_i{}^{p,\bar{p}}(x)$: Parton Distributions Functions (PDF)

($x \to$ fraction of hadron momentum carried by parton i)

$\longrightarrow$ $d\hat{\sigma}(ij \to X)$: partonic cross section

$\longrightarrow$ μ_F: factorization scale.

$\longrightarrow$ Q^2: hard scattering scale.

$\longrightarrow$ In the $ij \to X$ process, initial and final state partons radiate and absorb gluons/quarks (both real and virtual).

$\longrightarrow$ Due to the very same interactions: the strong coupling constant ($\alpha_s = g_s^2/4\pi$) becomes small at large energies (Q^2):

$$\alpha_s(Q^2) \to 0 \quad \text{for large scales } Q^2 \quad : \quad \text{asymptotic freedom}$$

$\longrightarrow$ We can calculate $d\hat{\sigma}(ij \to X)$ perturbatively:

$$d\hat{\sigma}(ij \to X) = \alpha_s^k \sum_{m=0}^{\infty} d\hat{\sigma}_{ij}^{(m)} \alpha_s^m$$

n=0 : Leading Order (LO), or tree level or Born level

n=1 : Next to Leading Order (NLO), includes $O(\alpha_s)$ corrections

......

$\longrightarrow$ We can calculate the evolution of the PDF's perturbatively (from DGLAP equation): LO, NLO, ... PDF's.

Perturbative approach and scale dependence ...

$\longrightarrow$ At each order in α_s both partonic cross section and PDF's have a residual factorization scale dependence (μ_F).

$\longrightarrow$ At each order in α_s the expression of $\hat{\sigma}(ij \to X)$ contains UV infinities that are renormalized. A remnant of the subtraction point is left at each perturbative order as a renormalization scale dependence (μ_R)

$$d\hat{\sigma}(ij \to X, Q^2, \mu_F) = \alpha_s^k(\mu_R) \sum_{m=0}^{n} d\hat{\sigma}_{ij}^{(m)}(Q^2, \mu_F, \mu_R)\alpha_s^m(\mu_R)$$

Setting $\boxed{\mu_R = \mu_F = \mu}$ (often adopted simplifying assumption):

$$d\sigma = \sum_{ij} \int dx_1 dx_2 f_i^p(x_1, \mu) f_j^{p,\bar{p}}(x_2, \mu) \sum_{m=0}^{n} d\hat{\sigma}_{ij}^{(m)}(x_1, x_2, Q^2, \mu)\alpha_s^{m+k}(\mu)$$

The residual scale dependence should improve with the perturbative order

General structure of a NLO calculation

NLO cross-section:

$$d\sigma^{NLO}_{p\bar{p},pp} = \sum_{i,j} \int dx_1 dx_2 f_i^p(x_1,\mu) f_j^{\bar{p},p}(x_2,\mu) d\hat{\sigma}^{NLO}_{ij}(x_1,x_2,\mu)$$

where

$$d\hat{\sigma}^{NLO}_{ij} = d\hat{\sigma}^{LO}_{ij} + \frac{\alpha_s}{4\pi} \delta d\hat{\sigma}^{NLO}_{ij}$$

NLO corrections made of:

$$\delta d\hat{\sigma}^{NLO}_{ij} = d\hat{\sigma}^{virt}_{ij} + d\hat{\sigma}^{real}_{ij}$$

- $d\hat{\sigma}^{virt}_{ij}$: one loop virtual corrections.
- $d\hat{\sigma}^{real}_{ij}$: one gluon/quark real emission.
- use $\alpha_s^{NLO}(\mu)$ and match with NLO PDF's.

$\longrightarrow$ renormalize UV divergences ($d=4-2\epsilon_{UV}$)

$\longrightarrow$ cancel IR divergences in $d\hat{\sigma}^{virt}_{ij} + d\hat{\sigma}^{real}_{ij}$ + PDF's ($d=4-2\epsilon_{IR}$)

$\longrightarrow$ check μ-dependence of $d\sigma^{NLO}_{p\bar{p},pp}$ (μ_R,μ_F)

On the issue of scale dependence ...

At a given order, the dependence on the renormalization/factorization scales is a higher order effect.

Let us rewrite $d\hat{\sigma}_{ij}^{NLO}(x_1, x_2, \mu)$ making the scale dependence explicit:

$$d\hat{\sigma}_{ij}^{NLO}(x_1, x_2, \mu) = \alpha_s^k(\mu) \left\{ F_{ij}^{LO}(x_1, x_2) + \frac{\alpha_s(\mu)}{4\pi} \left[\mathcal{F}_{ij}^1(x_1, x_2) + \bar{\mathcal{F}}_{ij}^1(x_1, x_2) \ln\left(\frac{\mu^2}{\hat{s}}\right) \right] \right\}$$

$\bar{\mathcal{F}}_{ij}^1(x_1, x_2)$ can be calculated by imposing that the hadronic cross-section is scale independent at the perturbative order of the calculation (NLO), i.e.:

$$\mu^2 \frac{d}{d\mu^2} d\sigma_{p\bar{p},pp}^{NLO} = \mathcal{O}(\alpha_s^{(k+2)})$$

Using that $\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu) = -b_0 \alpha_s^2 + \cdots$, and the DGLAP equation for the scale evolution of PDF's, one gets:

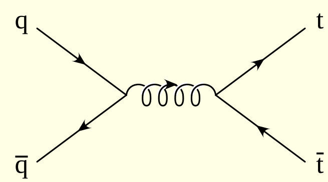
$$\begin{aligned} \bar{F}_{ij}^1(x_1, x_2) &= 2 \left\{ 4\pi b_0 F_{ij}^{LO}(x_1, x_2) \right. \\ &- \left. \sum_k \left[\int_\rho^1 dz_1 P_{ik}(z_1) F_{kj}^{LO}(x_1 z_1, x_2) + \int_\rho^1 dz_2 P_{jk}(z_2) F_{ik}^{LO}(x_1, x_2 z_2) \right] \right\} \end{aligned}$$

($P_{ij} \longrightarrow$ Altarelli-Parisi splitting functions).

Example: $p\bar{p}, pp \longrightarrow t\bar{t}$, tree level

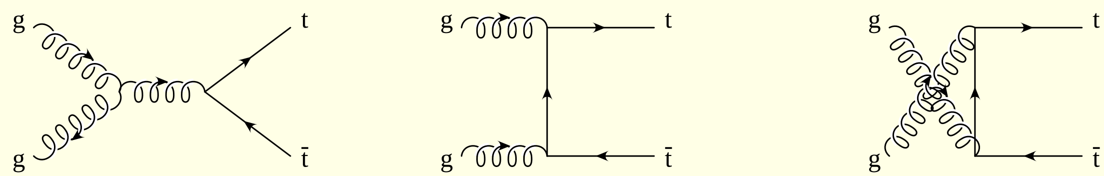
$q\bar{q} \longrightarrow t\bar{t}$

leading contribution at the Tevatron



$gg \longrightarrow t\bar{t}$

leading contribution at the LHC



NLO corrections calculated in:

→ P. Nason, S. Dawson, R.K. Ellis, NPB 303 (1988) 607, NPB 327 (1989) 49;

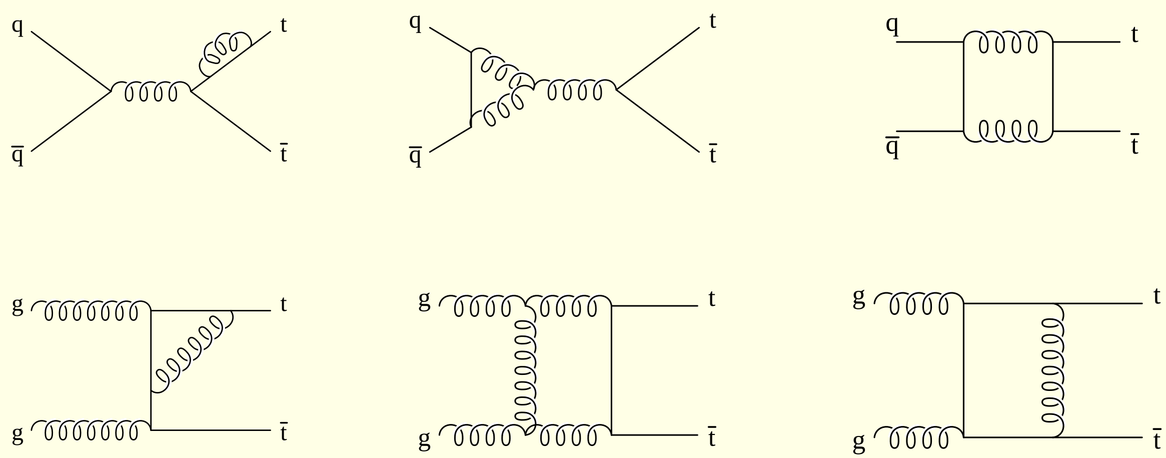
→ W. Beenakker, H. Kuijf, W.L. van Neerven, J. Smith PRD 40 (1989) 54; (with R. Meng and G.A. Schuler) NPB 351 (1991) 507.

$p\bar{p}, pp \to t\bar{t}$: $\mathcal{O}(\alpha_s)$ virtual corrections

The $\mathcal{O}(\alpha_s)$ virtual corrections to the cross-section arise from the interference between the tree level amplitude $\mathcal{A}_0(q\bar{q}, gg \to t\bar{t})$ and the one-loop virtual amplitude $\mathcal{A}_1^{virt}(q\bar{q}, gg \to t\bar{t})$:

$$d\hat{\sigma}_{ij}^{virt} = d(PS_2)\, 2Re(\mathcal{A}_1^{virt}\mathcal{A}_0^*)$$

where $\mathcal{A}_1^{virt}(q\bar{q}, gg \to t\bar{t})$ receives contributions from self-energy, vertex and box type loop-corrections:

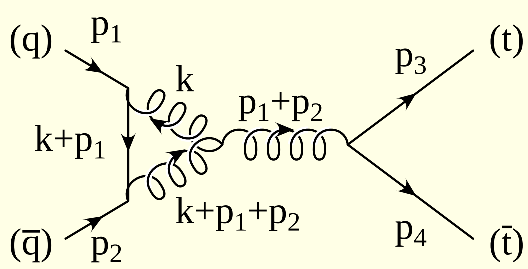


most of which contains UV and IR divergences that need to be extracted analytically.

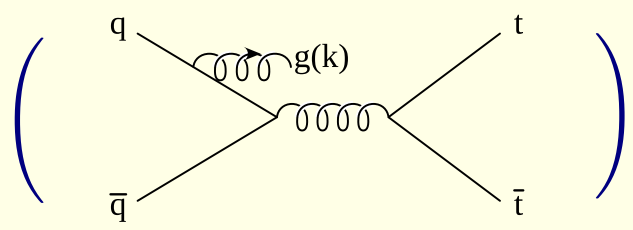
For each diagram:

- write the corresponding amplitude and reduce the Dirac's algebra to fundamental structures;
- the coefficient of each Dirac's structure contains both scalar and tensor integrals of the loop momentum;
- tensor integrals can be reduced to scalar integrals;
- scalar integrals are computed and UV and IR divergences are extracted using dimensional regularization in $d = 4 - 2\epsilon$;
- UV divergences (two- and three-point functions) are extracted as poles in $\frac{1}{\epsilon_{UV}}$, and subtracted using a given renormalization scheme ($\overline{MS}$, on-shell, etc.);
- IR divergences (two-, three- and four-point functions) are extracted as poles in $\frac{1}{\epsilon_{IR}^2}$ and $\frac{1}{\epsilon_{IR}}$ $\longrightarrow$ cancelled in $d\hat{\sigma}_{ij}^{virt} + d\hat{\sigma}_{ij}^{real} +$ PDF's.

Example. Consider the vertex correction:



$\longleftrightarrow$



the amplitude associated to this diagram is of the form:

$$\mathcal{A} \propto \int \frac{d^dk}{(2\pi)^d} \frac{\bar{v}(p_2)\gamma_\rho(\not{k}+\not{p}_1)\gamma_\nu u(p_1)}{k^2(k+p_1)^2(k+p_1+p_2)^2} \frac{\bar{u}(p_3)\gamma_\mu v(p_4)}{(p_1+p_2)^2} V^{\mu\nu\rho}(-p_1-p_2,-k,k+p_1+p_2)$$

and depends on the following scalar/tensor integrals:

$$C_0, C_1^\mu, C_2^{\mu\nu}(p_1,p_2) = \int \frac{d^dk}{(2\pi)^d} \frac{1, k^\mu, k^\mu k^\nu}{k^2(k+p_1)^2(k+p_1+p_2)^2} = \int \frac{d^dk}{(2\pi)^d} \frac{1, k^\mu, k^\mu k^\nu}{D_3(p_1,p_2)}$$

$C_2^{\mu\nu}$ is UV divergent, while all of them are IR divergent, as can be easily recognized by simple power counting and observing that

$$D_3(p_1,p_2) \overset{k\to k-p_1}{\longrightarrow} k^2(k-p_1)^2(k+p_2)^2 \longrightarrow k^2(k\cdot p_1)(k\cdot p_2)$$

- $k^0 \to 0:$ soft divergence;
- $k\cdot p_1 \to 0$ or $k\cdot p_2 \to 0$: collinear divergence.

We can parametrize C_1^μ and $C_2^{\mu\nu}$ as:

$$
\begin{aligned}
C_1^\mu(p_1,p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{D_3(p_1,p_2)} &= C_1^1 p_1^\mu + C_1^2 p_2^\mu \\
C_2^{\mu\nu}(p_1,p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{D_3(p_1,p_2)} &= C_2^{00} g^{\mu\nu} + C_2^{11} p_1^\mu p_1^\nu + C_2^{22} p_2^\mu p_2^\nu + \\
& \quad C_2^{12}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)
\end{aligned}
$$

The tensor integral coefficients can be obtained using different methods. In this case the easiest way is by saturating the independent tensor structures: Passarino-Veltman method (Nucl. Phys. B 160 (1979) 151).

Ex: C_1^1 and C_1^2 are very simple:

$$
\begin{aligned}
C_1^1 &= \frac{1}{2p_1\cdot p_2}\left(B0_{12} - B0_{13} - 2p_1\cdot p_2 C_0\right) \\
C_1^2 &= \frac{1}{2p_1\cdot p_2}\left(B0_{13} - B0_{23}\right)
\end{aligned}
$$

where $B0_{ij}$ are scalar integrals with two (out of the three) denominators of C_0.

Ultimately, everything is expressed in terms of B_0, and C_0 scalar integrals.

Introducing the appropriate Feynman's parameterization the diagram in question can be written as:

$$\Gamma(3)\int_0^1 dx\, x\int_0^1 dy\int\frac{d^dk'}{(2\pi)^d}\frac{\mathrm{Num}(k\to k')}{[(k')^2-\Delta]^3}$$

where $(k')^\mu = (k - x(1-y)p_1 + (1-x)p_2)^\mu$ and $\Delta = -x(1-x)(1-y)2p_1\cdot p_2$.

Using standard d-dimensional integrals:

$$\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\frac{1}{\Delta^{n-d/2}}$$

and upon integration over the Feynman parameters one gets (including couplings and color factor):

$$\frac{\alpha_s}{4\pi}\left(\frac{4\pi\mu}{\hat{s}}\right)^\epsilon\frac{N}{2}\Gamma(1+\epsilon)\left(-\frac{4}{\epsilon_{IR}}+\frac{3}{\epsilon_{UV}}-2\right)\mathcal{A}_0(q\bar{q}\to t\bar{t})$$

This is a very simple case, but more complex ones are solved using the same strategy.

Passarino-Veltman reduction can be problematic when considering high rank tensor integrals in processes with more external particles.
The tensor integrals coefficients are proportional to inverse powers of the Gram determinant (GD):

$$GD = \det(p_i \cdot p_j) \qquad (p_i, p_j \rightarrow \text{independent external momenta})$$

(where $GD = p_1 \cdot p_2$ in the example we saw.) For instance, for a $2 \rightarrow 3$ process:

$$GD(p_1 + p_2 \rightarrow p_3 + p_4 + p_5) \simeq f(E_3, E_4, \sin\theta_3, \sin\theta_4, \sin\phi_4)$$

$\boxed{GD \rightarrow 0}$ when two momenta become degenerate: <u>spurious divergences</u> that creates <u>numerical instabilities</u>.

<u>Possible alternatives</u> (more in next lecture):

$\longrightarrow$ Eliminate all dangerous tensor integrals at the level of the amplitude square, if possible.

$\longrightarrow$ Kinematic cuts to avoid numerical instabilities and extrapolation to the unsafe region using several algorithms.

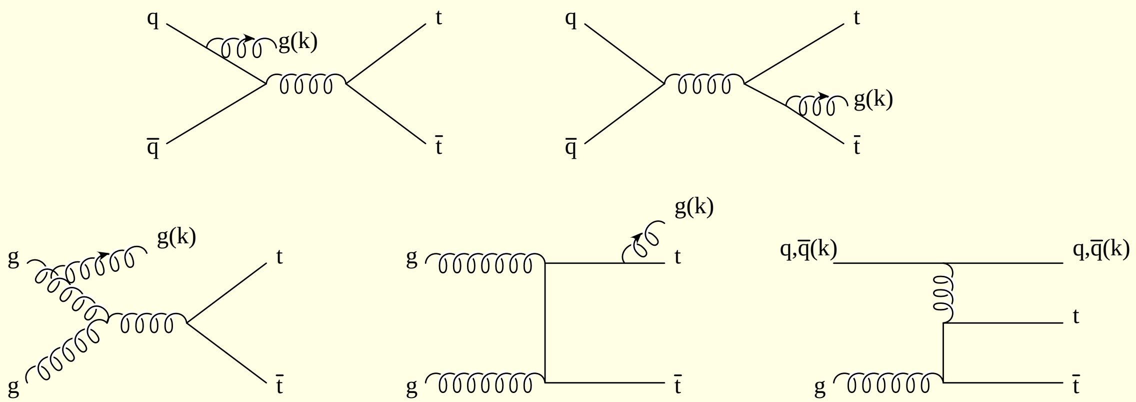
$\longrightarrow$ Expansion of coefficients about limit of vanishing kinematic determinants, including $GD \rightarrow 0$ (S. Dittmaier, A. Denner, NPB 734 (2006) 62)

$p\bar{p}, pp \to t\bar{t}$: $\mathcal{O}(\alpha_s)$ real corrections

The $\mathcal{O}(\alpha_s)$ real corrections to the cross-section arise from the square of the real gluon/quark emission amplitude $\mathcal{A}_1^{real}(q\bar{q}, gg, qg \to t\bar{t} + (g/q/\bar{q}))$:

$$d\hat{\sigma}_{ij}^{real} = d(PS_{2+(g/q/\bar{q})})|\mathcal{A}_1^{real}|^2$$

where $\mathcal{A}_1^{real}$ receives contributions from diagrams like:



IR singularities are extracted by looking for the region of the $t\bar{t} + (g/q/\bar{q})$ phase space where $s_{ik} \to 0$, where:

$$s_{ik} = 2p_i \cdot k = 2E_i k^0 (1 - \beta_i \cos\theta_{ik})$$

- $k^0 \to 0$: soft singularities (both massless and massive particles);
- $\cos\theta_{ik} \to 0$: collinear singularities (massless particles only).

Soft and collinear singularities can be isolated thanks to the factorization properties of $|\mathcal{A}_1^{real}|^2$ and $d(PS_{2+(g/q/\bar{q})})$ in the soft and collinear limits.

Consider $\boxed{ij \to t\bar{t} + g}$. In the soft limit $(E_g = k^0 \to 0)$:

$$d(PS_{2+g}) \xrightarrow{soft} d(PS_2)d(PS_g) = d(PS_2)\frac{d^{d-1}k}{(2\pi)^{d-1}2k^0}$$

$$|\mathcal{A}_1^{real}(ij \to t\bar{t} + g)|^2 \xrightarrow{soft} (4\pi\alpha_s)|\mathcal{A}_0|^2\Phi_{eik}$$

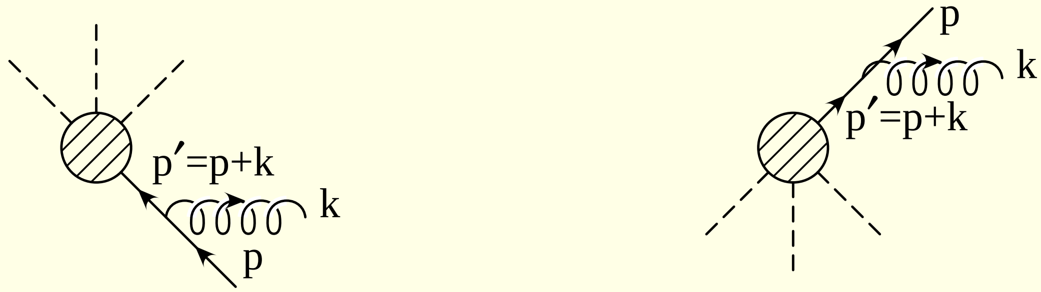
where the eikonal factor Φ_{eik} contains the soft poles $(s_{ij} = (p_i + p_j)^2)$:

$$\Phi_{eik} \propto \sum_{ij}\left(\frac{s_{ij}}{s_{ig}s_{jk}} - \frac{m_i^2}{s_{ik}^2} - \frac{m_j^2}{s_{jk}^2}\right)$$

$$\Downarrow$$

$$d\hat{\sigma}_{ij}^{soft} \propto d(PS_2)\int_{soft} d(PS_g)\Phi_{eik}|\mathcal{A}_0|^2$$

Soft limit, a closer look …



Calculate $\mathcal{A}_{n+1}$ using:

- incoming line:

$$\cdots \frac{\not{p}' + m}{[(p')^2 - m^2]} \gamma^\mu u(p) \epsilon^*_\mu(k) \xrightarrow{k \to 0} \cdots \frac{\not{p} + m}{2p \cdot k} \gamma^\mu u(p) \epsilon^*_\mu(k) = -i g_s T^a \frac{p \cdot \epsilon^*(k)}{p \cdot k} \mathcal{A}_n$$

- outgoing line:

$$\bar{u}(p) \gamma^\mu \frac{\not{p}' + m}{[(p')^2 - m^2]} \epsilon^*_\mu(k) \ldots \xrightarrow{k \to 0} \bar{u}(p) \gamma^\mu \frac{\not{p} + m}{2p \cdot k} \epsilon^*_\mu(k) \ldots = -i g_s T^a \frac{p \cdot \epsilon^*(k)}{p \cdot k} \mathcal{A}_n$$

When squaring A_{n+1} the eikonal factor appears:

$$|\mathcal{A}_{n+1}| = g_s^2 C_F \left| \sum_i \frac{p_i \cdot \epsilon^*(k)}{p_i \cdot k} \right|^2 |\mathcal{A}_n|^2 = (4\pi\alpha_s) \Phi_{eik} |\mathcal{A}_n|^2$$

In the collinear limit ($i \to i'g$, $p'_i = z p_i$, $k = (1-z) p_i$)

$$|\mathcal{A}_1^{real}(ij \to t\bar{t} + g)|^2 \stackrel{collinear}{\longrightarrow} (4\pi\alpha_s)|\mathcal{A}_0(i'j \to t\bar{t})|^2 \frac{2P_{ii'}(z)}{z\, s_{ik}}$$

$$d(PS_{2+g})(ij \to t\bar{t}) \stackrel{collinear}{\longrightarrow} d(PS_2)(i'j \to t\bar{t}) z d(PS_g)$$

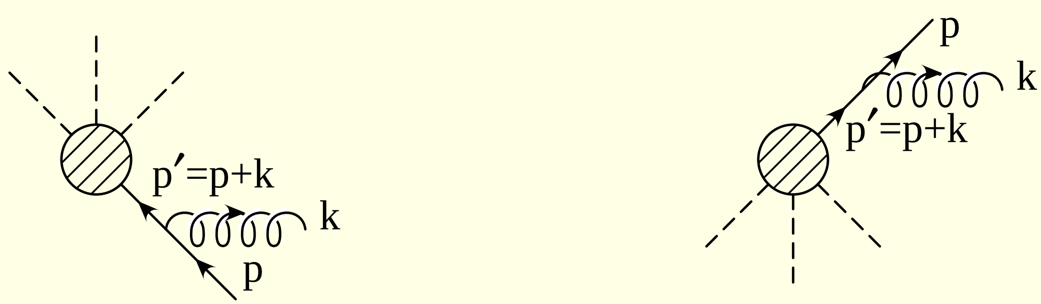
($P_{ii'} \to$ Altarelli-Parisi splitting functions)

$$\Downarrow$$

$$d\hat{\sigma}_{ij}^{hard/coll} \propto d(PS_2) \int_{coll} d(PS_g) \sum_i \frac{P_{ii'}}{s_{ik}} |\mathcal{A}_0(i'j \to t\bar{t})|^2$$

The idea is now to calculate analytically only the singular parts of σ_{ij}^{real}, while integrating numerically over the regions of the final state phase space that do not contain singularities. Even more so when we think of calculating processes with several particles (some of which massive) in the final state.

Collinear limit, a closer look ...



Use collinear kinematics, imposing $k^2 = 0$ (i.e. $k^2 = \mathcal{O}(p_\perp^4)$):

$$p' = \left(zp, -p_\perp, 0, zp + \frac{p_\perp^2}{2(1-z)p} \right)$$

$$k = \left((1-z)p, p_\perp, 0, (1-z)p - \frac{p_\perp^2}{2(1-z)p} \right)$$

and calculate $|\mathcal{A}_{n+1}^{real}|^2$ using that $(p')^2 = p_\perp^2/(1-z)$, to obtain:

$$|\mathcal{A}_{n+1}^{real}|^2 \stackrel{collinear}{\longrightarrow} (4\pi\alpha_s) \frac{2P_{qq}(z)}{z\, s_{pk}} |\mathcal{A}_n|^2$$

where $s_{pk} = 2p \cdot k$ and P_{qq} is $q \to q$ splitting function.

IR singularities are extracted:

- by imposing suitable cuts on the phase space of the radiated parton: phase space slicing (PSS) method.
  - $\longrightarrow$ PSS with two cutoffs:

    B.W. Harris and J.F. Owens, PRD 65 (2002) 094032 (review paper);
  - $\longrightarrow$ PSS with one cutoff:

    W.T. Giele, E.W.N. Glover and D.A. Kosower, PRD 46 (1992) 1980; NPB 403 (1993) 633; S. Keller and E. Laenen, PRD 59 (1999) 114004.
- by using a subtraction method:
  - R.K. Ellis, D.A. Ross, A.E. Terrano, NPB 178 (1981) 421;
  - S. Catani, S. Dittmaier, M.H. Seymour and Z. Trócsányi, NPB 627 (2002) 189 (and references therein).

Remaining initial-state IR singularities are absorbed in the PDF's (mass factorization).

PSS: Two Cutoff Method (δ_s,δ_c):

$$d\hat{\sigma}_{ij}^{real}(ij \to t\bar{t} + g) = d\hat{\sigma}_{ij}^{soft} + d\hat{\sigma}_{ij}^{hard/coll} + d\hat{\sigma}_{ij}^{hard/non-coll}$$

where

- $d\hat{\sigma}_{ij}^{soft} \longrightarrow E_g < \frac{\sqrt{s}}{2}\delta_s$
- $d\hat{\sigma}_{ij}^{hard/coll} \longrightarrow E_g > \frac{\sqrt{s}}{2}\delta_s$ and $(1 - \cos\theta_{ik}) < \delta_c$

are computed analytically to extract the IR singularities, while:

- $d\hat{\sigma}_{ij}^{hard/non-coll} \longrightarrow E_g > \frac{\sqrt{s}}{2}\delta_s$ and $(1 - \cos\theta_{ik}) > \delta_c$

can be computed numerically, since is IR finite.

$$\Downarrow$$

The dependence on the cutoffs needs to cancel in the phsyical cross section.

PSS: One Cutoff Method (s_{min}):

$$d\hat{\sigma}_{ij}^{real}(ij \to t\bar{t} + g) = \hat{\sigma}_{ij}^{ir} + \hat{\sigma}_{ij}^{hard}$$

where

- $d\hat{\sigma}_{ij}^{ir} \longrightarrow s_{ik} < s_{min}$

is computed analytically to extract the IR singularities:
$\longrightarrow$ cross all colored particles to final state;
$\longrightarrow$ work with color ordered amplitudes: easier matching between soft and collinear region;
$\longrightarrow$ introduce crossing functions: to account for difference between initial state and final state collinear singularities.

- $d\hat{\sigma}_{ij}^{hard} \longrightarrow s_{ig} > s_{min}$

can be computed numerically, since IR finite.

$$\Downarrow$$

The dependence on the cutoff needs to cancel in the phsyical cross section.

Consider e.g. $ij \to t\bar{t} + g$:

$$\hat{\sigma}_{ij}^{real} = \int d(PS_5) \overline{\sum} |\mathcal{A}_{ijt\bar{t}+g}^{real}|^2 \ ,$$

where

$$\mathcal{A}_{ijt\bar{t}+g}^{real} = \sum_{\substack{a,b,c \\ i \neq j \neq k}} \mathcal{A}_{abc} T^a T^b T^c \ .$$

$T^a \to$ color matrices, $\mathcal{A}_{abc} \to$ color ordered amplitudes.

The amplitude square is made of three terms:

$$\overline{\sum} |\mathcal{A}_{ijt\bar{t}+g}^{real}|^2 = \overline{\sum} \frac{(N^2-1)}{2} \left[\frac{N^2}{4} \sum_{\substack{a,b,c \\ a \neq b \neq c}} |\mathcal{A}_{abc}|^2 \right.$$

$$\left. - \frac{1}{4} \sum_{\substack{a,b,c \\ a \neq b \neq c}} |\mathcal{A}_{abc} + \mathcal{A}_{acb} + \mathcal{A}_{cab}|^2 + \frac{1}{4} \left(1 + \frac{1}{N^2}\right) \left| \sum_{\substack{a,b,c \\ a \neq b \neq c}} \mathcal{A}_{abc} \right|^2 \right]$$

with very definite soft/collinear factorization properties.

Soft singularities $\longrightarrow$ straightforward

How to disentangle soft vs collinear region of PS with one cutoff?

Collinear limit for $ig \to i'$: $s_{ig} \to 0$ $(i{=}g_1, g_2)$ $\begin{cases} p_i = z p'_i \\ p_g = (1-z) p'_i \end{cases}$

Each $\mathcal{A}_{ijk}$ (or linear combination of) proportional to $(s_{ai} s_{ig} s_{gb})^{-1}$

collinear region $\begin{cases} s_{ig} < s_{min} \\ s_{ai} > s_{min} \longrightarrow z s_{ai'} > s_{min} \longrightarrow z > z_1 = \frac{s_{min}}{s_{ai'}} \\ s_{gb} > s_{min} \longrightarrow (1-z) s_{i'b} > s_{min} \longrightarrow z < 1 - z_2 = 1 - \frac{s_{min}}{s_{i'b}} \end{cases}$

$z_1,\ 1 - z_2 \;\longrightarrow\;$ integration boundaries

How to match with $\hat{\sigma}_{hard}$? Match each term in $|\mathcal{A}^{real}_{ijt\bar{t}+g}|^2$ separately.

Subtraction Method

Subtract the singular behavior without introducing cutoffs. Schematically:

$$d\hat{\sigma}_{ij}^{NLO} = \left[d\hat{\sigma}_{ij}^{real} - d\hat{\sigma}_{ij}^{sub}\right]_{\epsilon\rightarrow 0} + \left[d\hat{\sigma}_{ij}^{virt} + d\hat{\sigma}_{ij}^{sub,CT}\right]_{\epsilon\rightarrow 0}$$

where

- $d\hat{\sigma}_{ij}^{sub}$ has the same singular behavior as $d\hat{\sigma}_{ij}^{real}$ at each phase space point (in d dimensions);
- $d\hat{\sigma}_{ij}^{sub}$ has to be analytically integrable over the singular one-parton phase space in d dimensions, such that we can define the subtraction "counterterm":

$$d\hat{\sigma}_{ij}^{sub,CT} = \int d(PS_g)\, d\hat{\sigma}_{ij}^{sub}$$

In this way:

- $[d\hat{\sigma}_{ij}^{real} - d\hat{\sigma}_{ij}^{sub}]$ is integrable over the entire phase space, and the limit $\epsilon \longrightarrow 0$ can safely be taken;
- $[d\hat{\sigma}_{ij}^{virt} + d\hat{\sigma}_{ij}^{sub,CT}]$ is finite and integrable in $d = 4$ because (modulus the IR singularities that are factored in the renormalized PDF's) $d\hat{\sigma}_{ij}^{sub,CT}$ contains all the IR poles of $d\hat{\sigma}_{ij}^{virt}$.

$d\hat{\sigma}_{ij}^{sub}$ can be built using the so called dipole formalism.

(S. Catani and M.H. Seymour, NPB 485 (1997) 291)

"Dipoles" $\longrightarrow$ soft or collinear singular structures (dV_{dipole}) defined by the factorization of the real emission amplitude (see this lecture).

$$d\hat{\sigma}_{ij}^{sub} = \sum_{dipoles} d\hat{\sigma}_{ij}^{LO} \otimes dV_{dipole}$$

$d\hat{\sigma}_{ij}^{LO} \longrightarrow$ tree level cross-section.

such that:

$$d\hat{\sigma}_{ij}^{sub,CT} = \int d(PS_g)\, d\hat{\sigma}_{ij}^{sub} = d\hat{\sigma}_{ij}^{LO} \otimes \sum_{dipoles} \int d(PS_g)\, dV_{dipole}$$

What do we gain?

- Stability and predictivity of theoretical results, since less sensitivity to unphysical renormalization/factorization scales. First reliable normalization of total cross-sections and distributions. Crucial to:
  $\longrightarrow$ precision measurements (M_W, m_t, M_H, $y_{b,t}$, $\ldots$);
  $\longrightarrow$ searches for new physics (precise modeling of signal and background);
  $\longrightarrow$ reducing systematic errors in selection/analysis of data.

- Physics richness: more partons in final state, i.e. more structure to better model (in perturbative region):
  $\longrightarrow$ differential cross-sections, exclusive observables;
  $\longrightarrow$ jet formation/merging and hadronization;
  $\longrightarrow$ initial state radiation.

- First step towards matching with algorithms that resums particular set of large corrections in the perturbative expansion: resummed calculations, parton showers Monte Carlo's.

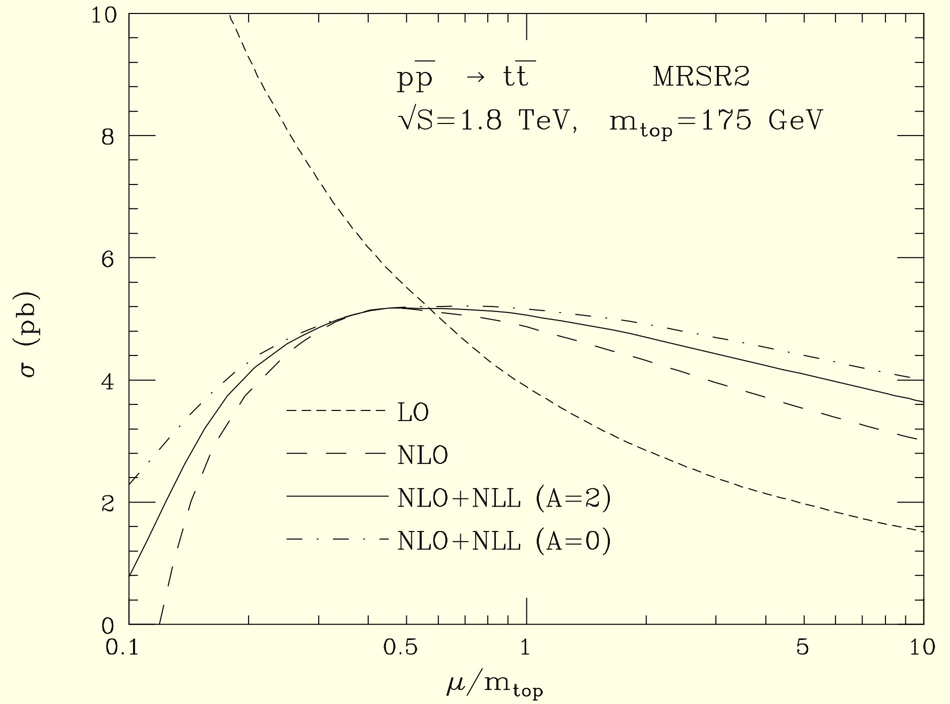
When is NLO not enough?

- When NLO corrections are large, to tests the convergence of the perturbative expansion. This may happen when:
  - $\longrightarrow$ processes involve multiple scales, leading to large logarithms of the ratio(s) of scales;
  - $\longrightarrow$ new parton level subprocesses first appear at NLO;
  - $\longrightarrow$ ...
- When truly high precision is needed (very often the case!).
- When a really reliable error estimate is needed.

$\Downarrow$

See examples to follow.

Example 1: $\boxed{p\bar{p} \to t\bar{t}}$, very reduced scale dependence.



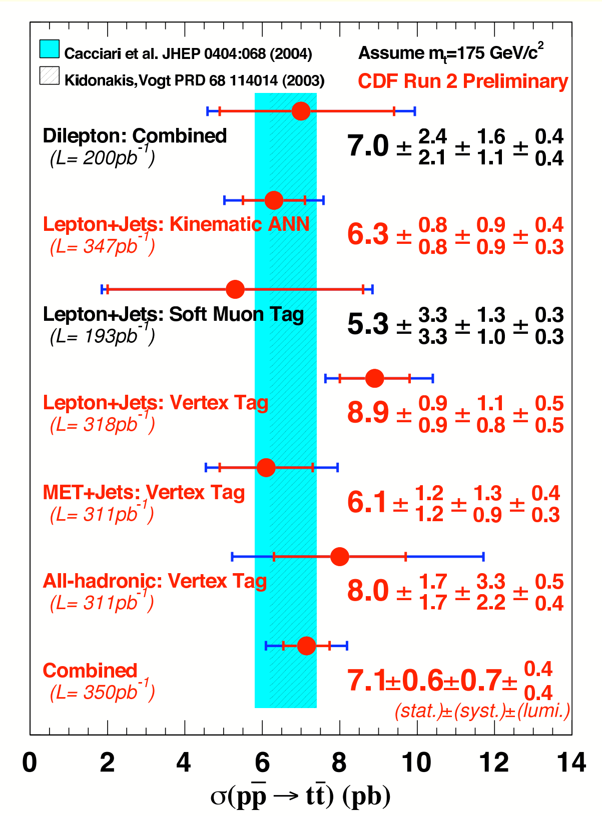
(R. Bonciani, S. Catani, M. Mangano, P. Nason, NPB 529 (1998) 424)

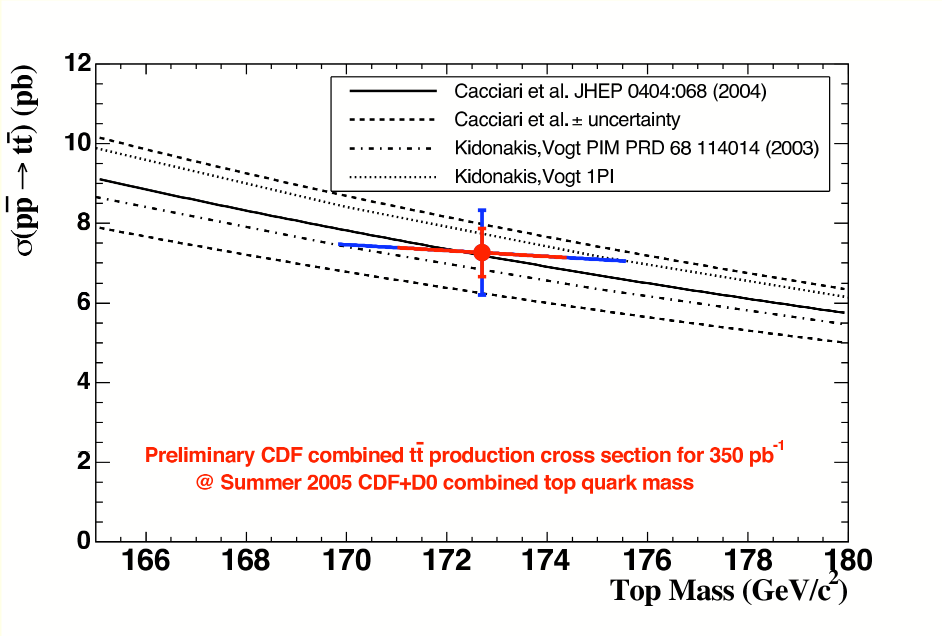
Tevatron: radiative corrections are large in the region near threshold ($\hat{s} = 4m_t^2$).
Calculation refined to resum higher order corrections due to soft gluon radiation ($\longrightarrow$ see C. Oleari's lectures).

NLO $\longrightarrow$ scale uncertainty $\simeq \pm 10\%$

NNL $\longrightarrow$ Next-to-Leading Logarithms, scale uncertainty $\simeq \pm 5\%$

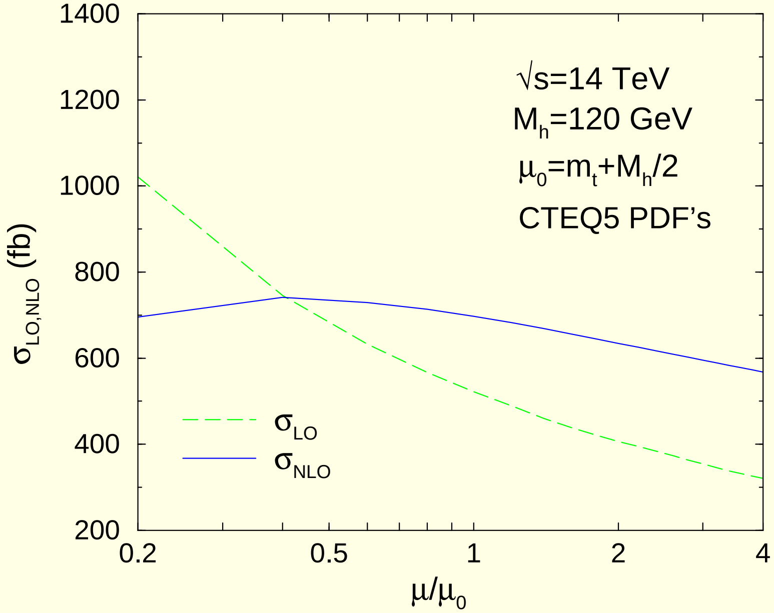
comparing to experimental results . . .

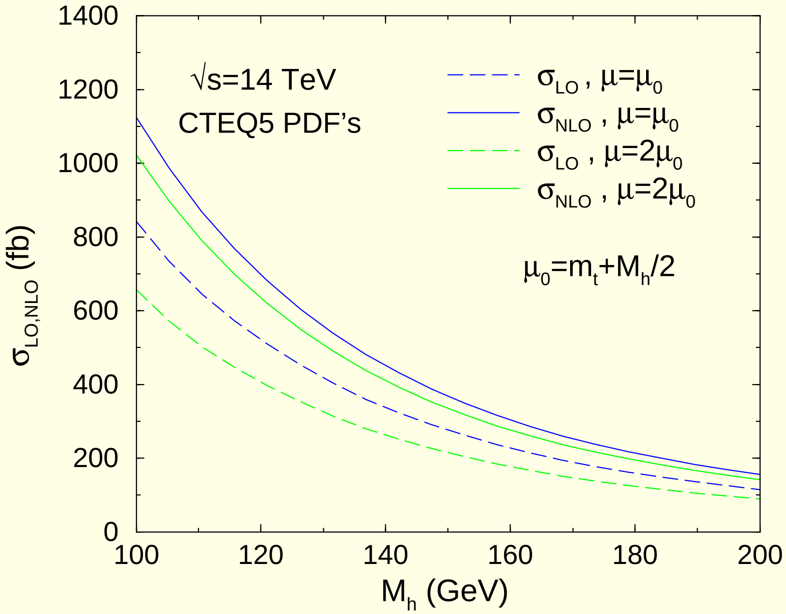




NLO and resummation of soft corrections crucial to match the $t\bar{t}$ cross-section measurement so closely. ($\longrightarrow$ see lecture on "Top quark at colliders")

Example 1: $\boxed{pp \to t\bar{t}H}$, very reduced scale dependence.





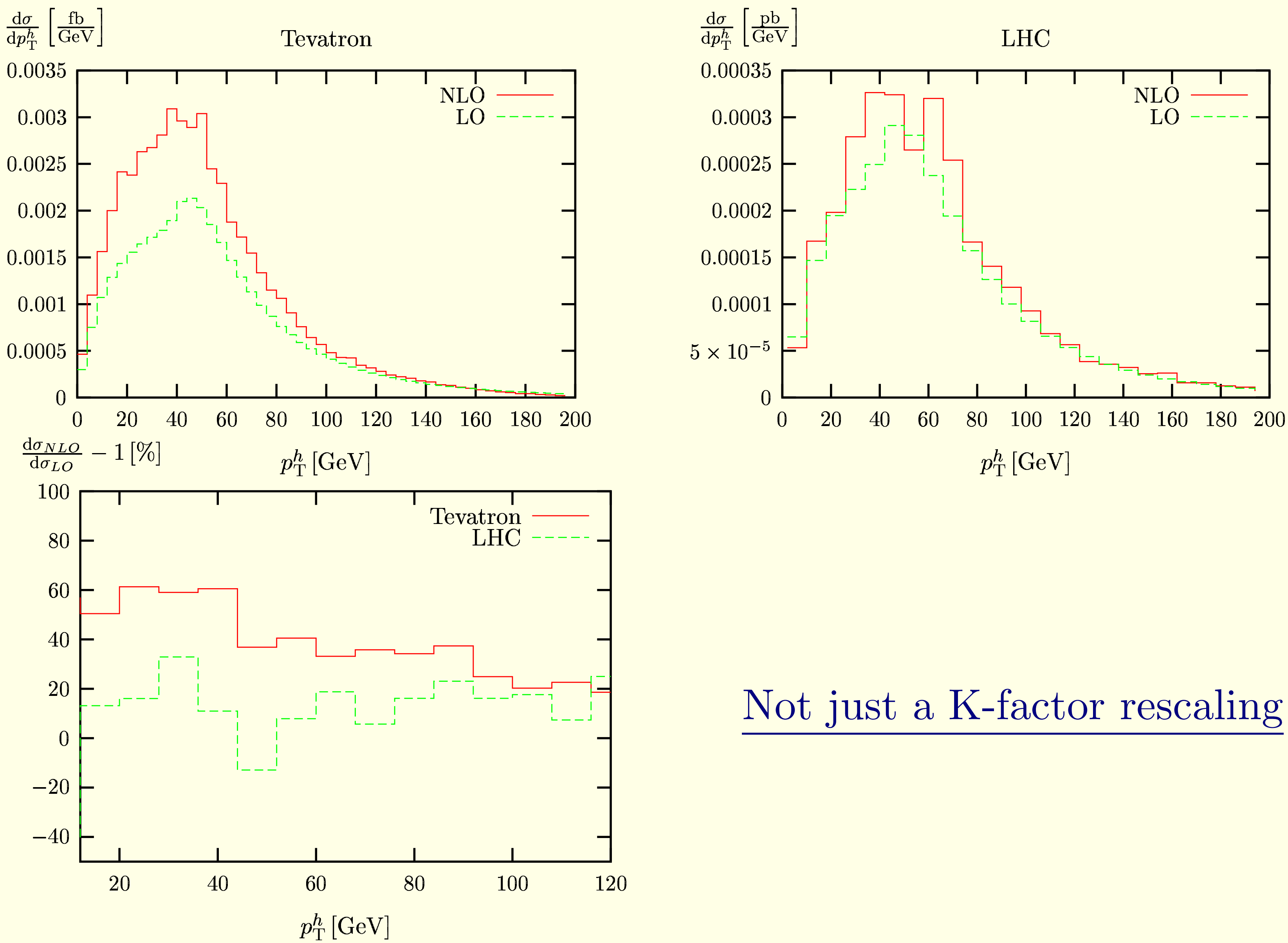
(S.Dawson, C.Jackson, L.H.Orr, L.R., D.Wackeroth, PRD 68 (2003) 034022)

Scale uncertainty reduced to about 15%

Higgs boson production with heavy quarks important both for discovery and to measure top/bottom Yukawa couplings (SM vs MSSM).

($\longrightarrow$ see lecture on "Top quark at colliders")

Example 3: $\boxed{p\bar{p}, pp \to b\bar{b}H}$, NLO distributions.

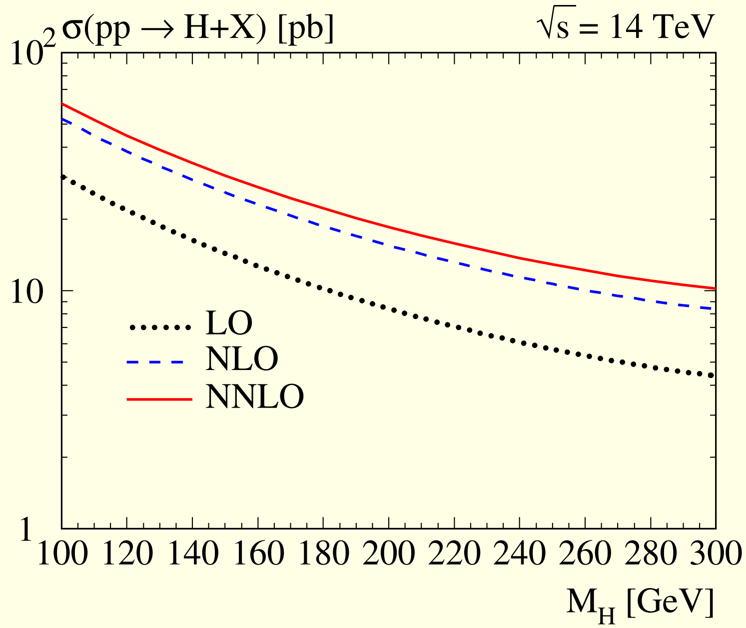


Not just a K-factor rescaling

(S.Dawson, C.Jackson, L.H.Orr, L.R., D.Wackeroth, PRD 69 (2004) 074027)

Example 4: $\boxed{gg \to H}$, stability at NNLO.

(R. Harlander, W. Kilgore, PRL 88 (2002) 201801)



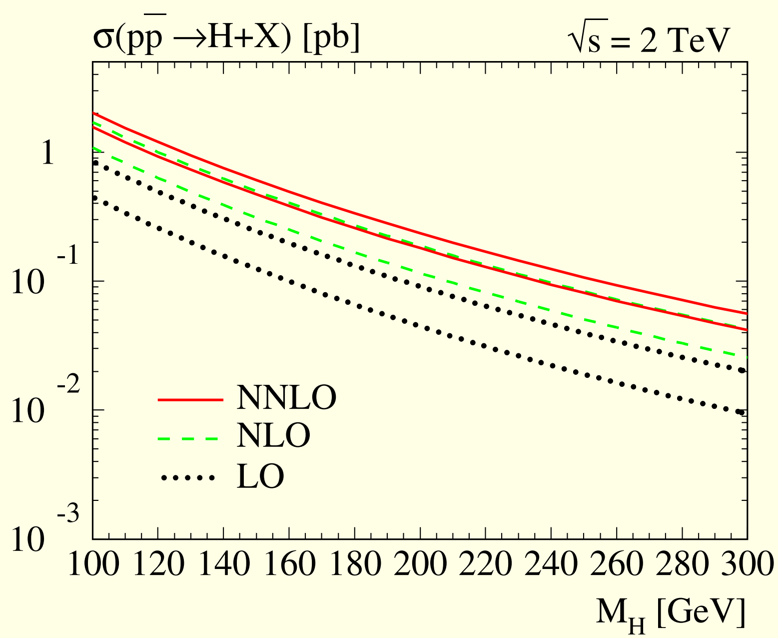
convergence in going:

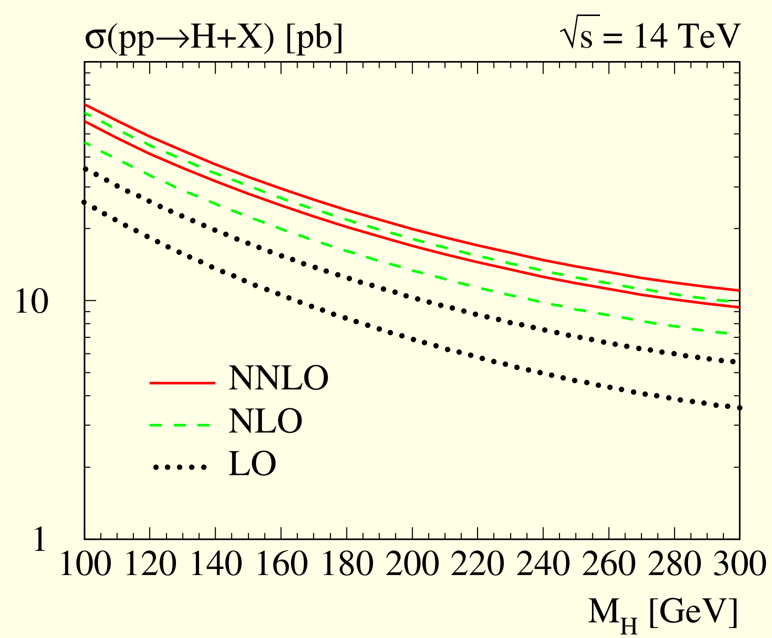
LO $\longrightarrow$ NLO $\longrightarrow$ NNLO

Confirmed by the full

scale dependence:

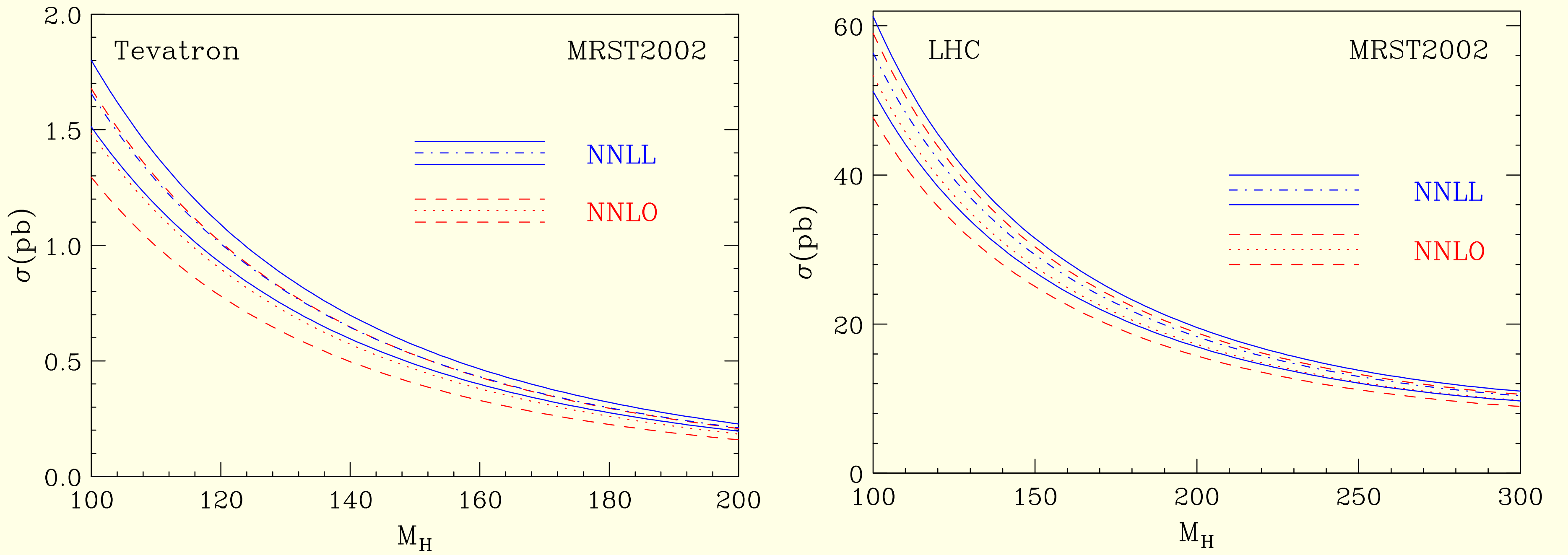
$\Downarrow$





Further improvement: resumming soft logarithms.

(→ see G. Sterman's lectures)



(S. Catani, D. de Florian, M. Grazzini, P. Nason, JHEP 0307 (2003) 028)

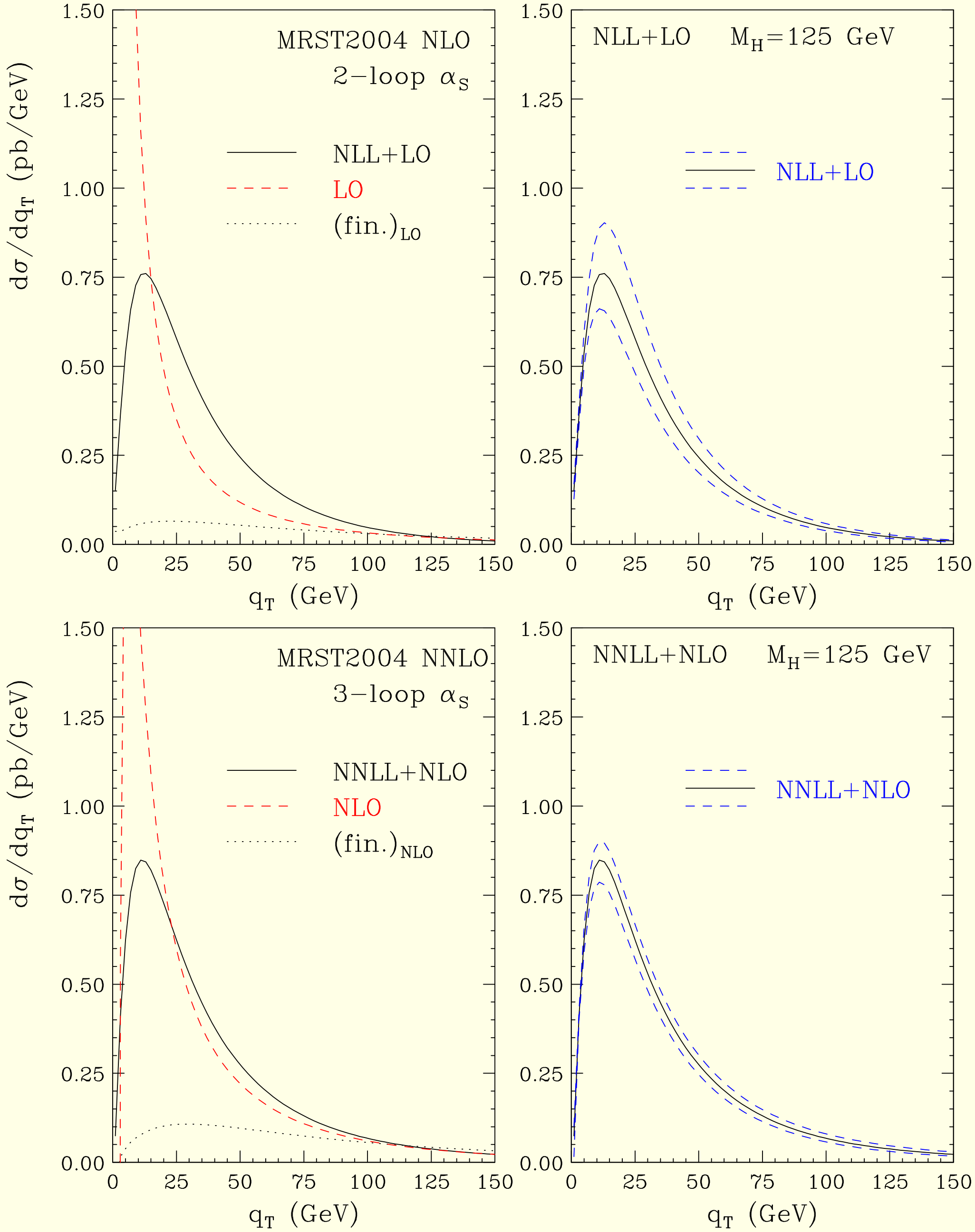
with **NNLO+NNLL** theoretical uncertainty reduced to:

$\longrightarrow$ $\simeq 10\%$ perturbative uncertainty, including the $m_t \to \infty$ approximation.

$\longrightarrow$ $\simeq 10\%$ from (now existing, but still to be tested) NNLO PDF's.

Resummation crucial in transverse momentum distributions.

(G. Bozzi, S. Catani, D. de Florian, M. Grazzini, NPB 737 (2006) 73)



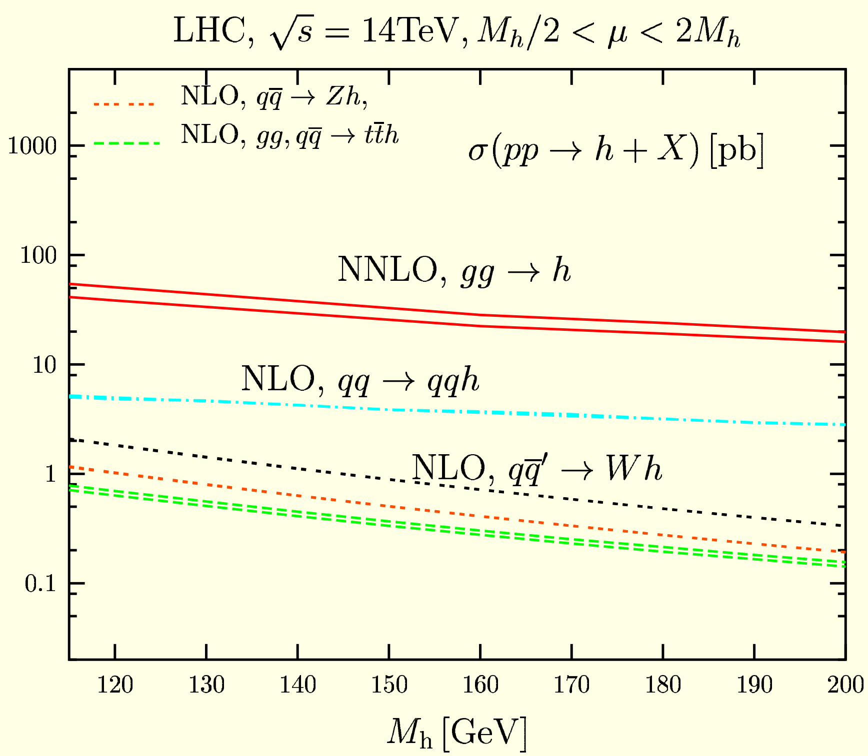
$q_T \longrightarrow$ Higgs boson transverse momentum

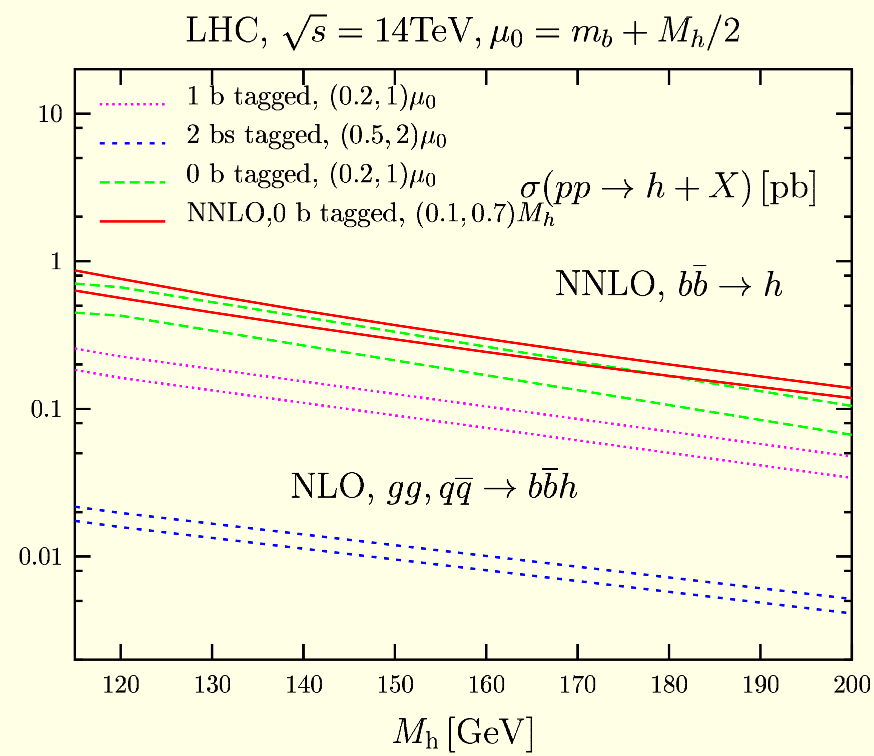
large $q_T \xrightarrow{q_T > M_H}$ perturbative expansion in $\alpha_s(\mu)$

small $q_T \xrightarrow{q_T \ll M_H}$ need to resum large $\ln(M_H^2/q_T^2)$

Example 5: Higgs production at the LHC, overview.

QCD predictions for total cross sections to Higgs production processes are under good theoretical control:





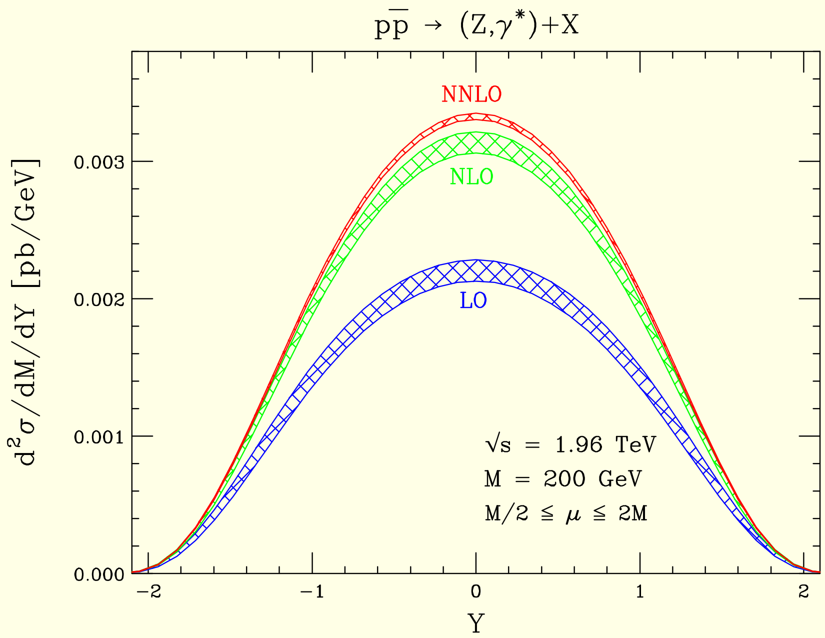
A word of caution:

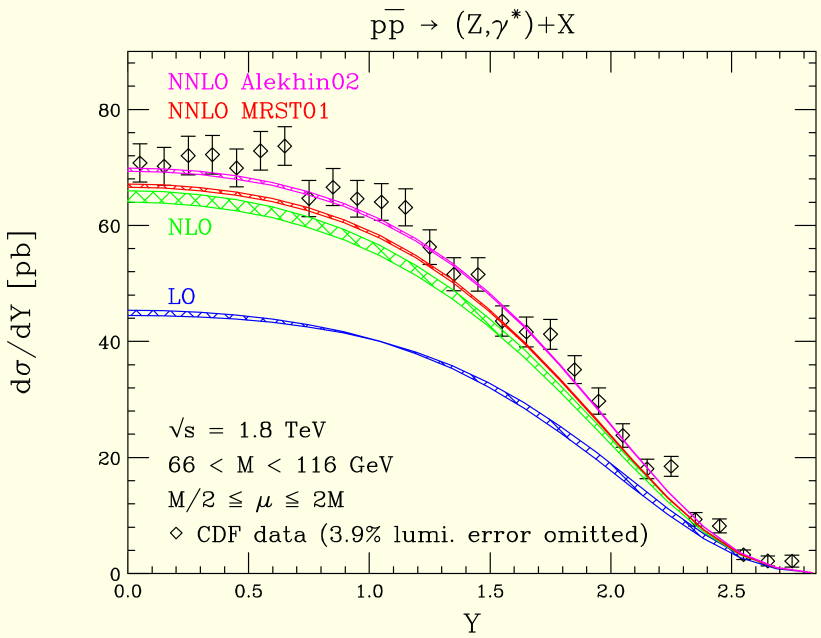
- ▷ uncertainties only include μ_R/μ_F dependence
- ▷ uncertainties from PDF's are not included (but should improve)

process	$\sigma_{NLO,NNLO}$ (by)
$gg \to H$	S.Dawson, NPB 359 (1991), A.Djouadi, M.Spira, P.Zerwas, PLB 264 (1991) C.J.Glosser *et al.*, JHEP (2002); V.Ravindran *et al.*, NPB 634 (2002) D. de Florian *et al.*, PRL 82 (1999) R.Harlander, W.Kilgore, PRL 88 (2002) (NNLO) C.Anastasiou, K.Melnikov, NPB 646 (2002) (NNLO) V.Ravindran *et al.*, NPB 665 (2003) (NNLO) S.Catani *et al.* JHEP 0307 (2003) (NNLL) G.Bozzi *et al.*, PLB 564 (2003), NPB 737 (2006) (NNLL)
$q\bar{q} \to (W,Z)H$	T.Han, S.Willenbrock, PLB 273 (1991) O.Brien, A.Djouadi, R.Harlander, PLB 579 (2004) (NNLO)
$q\bar{q} \to q\bar{q}H$	T.Han, G.Valencia, S.Willenbrock, PRL 69 (1992) T.Figy, C.Oleari, D.Zeppenfeld, PRD 68 (2003)
$q\bar{q}, gg \to t\bar{t}H$	W.Beenakker *et al.*, PRL 87 (2001), NPB 653 (2003) S.Dawson *et al.*, PRL 87 (2001), PRD 65 (2002), PRD 67,68 (2003)
$q\bar{q}, gg \to b\bar{b}H$	S.Dittmaier, M.Krämer, M.Spira, PRD 70 (2004) S.Dawson *et al.*, PRD 69 (2004), PRL 94 (2005)
$gb(\bar{b}) \to b(\bar{b})H$	J.Cambell *et al.*, PRD 67 (2003)
$b\bar{b} \to (b\bar{b})H$	D.A.Dicus *et al.* PRD 59 (1999); C.Balasz *et al.*, PRD 60 (1999). R.Harlander, W.Kilgore, PRD 68 (2003) (NNLO)

Example 6: W/Z production at the Tevatron, testing PDF's at NNLO.

Rapidity distributions of the Z boson calculated at NNLO:





(C. Anastasiou, L. Dixon, K. Melnikov, F. Petriello, PRL 91 (2003) 182002)

- W/Z production processes are standard candles at hadron colliders.
- Testing NNLO PDF's: parton-parton luminosity monitor, detector calibration.