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Black Hole Atoms

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BLACK HOLE ATOMS

By

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Abstract
In the earliest moments of the universe, we expect that there were microscopic density fluctuations arising from quantum processes. These fluctuations were made macroscopic by inflation and had the possibility of creating primordial black holes. We investigate the mass range at which these black holes would survive to today through analyzing their evaporation via Hawking radiation, and find that the minimum mass PBH that would survive to today from being created near the Big Bang is around 2×10^{11} kilograms. We also investigate the possibility of these black holes forming quantum bound states with electrons, and how those states compare to the standard Hydrogen atom model. We find, using the minimum mass, that the energy levels are somewhat altered from the Hydrogen model, overall forming slightly more tightly bound states.

1 Introduction

Primordial black holes (PBHs) are black holes that could have been created in the earliest moments of the formation of our Universe, arising from the slight density fluctuations created from quantum noise, the same noise that later led to the overall cosmological structure we see today [1]. The fluctuations are a requirement for the formation of these black holes since mass can only come together and form a singularity if there is a gradient in the density distribution, otherwise the mass would remain evenly distributed across all of space. While no PBHs have yet been observed, there is no known reason that would prevent their formation based on our understanding of the conditions in the early Universe. Their possible observation, however, turns out to be contingent on their mass. Steven Hawking, when considering quantum field theory in the weakly curved space far from a black hole, theorized in 1974 that black holes are not static objects when not absorbing matter, as the solutions to general relativity (GR) suggest, but instead emit particles in a way analogous to blackbody radiation [2]. This implies that black holes not only have a temperature associated with them, but also that they lose mass through this process, with their temperature (and hence evaporation rate) increasing as their mass decreases. Hawking Radiation, as it is known, therefore puts a lower limit on the mass of PBHs created in the early Universe that can still be observed. In the mass range associated with PBHs that would still exist today, their initial size would have been comparable to that of subatomic particles [3]. It appears entirely possible that, once formed, these PBHs could have attracted particles (such as electrons) in a bound state, resembling atoms with a similar electron orbital structure.

It will be these PBHs and their possible atom-like properties that will be studied in this thesis. PBHs have recently become subjects of analysis again due to their candidacy as dark matter particles (or at least a portion of them), as they would exhibit the same properties we infer dark matter to have, namely no or rare self interaction and no emission of any electromagnetic radiation. In order for them to be viable candidates, we need to know the initial lower mass limit of PBHs that would still exist today before totally evaporating. The first section of this thesis will seek to do this. Since the mass range has already been determined using similar methods to the ones presented in this section, it will serve to confirm those results. This mass range will be determined by analyzing how a black hole's mass changes over time from the effects of Hawking radiation and the absorption of cosmic microwave background (CMB) radiation throughout the lifetime of the Universe. Specifically, treating a black hole as a black body, the power emitted P as it thermally radiates can be determined using the Stefan-Boltzmann Law [4]:

$$P = A\sigma \left(T_{\mathrm{BH}}^4 - T_{\mathrm{CMB}}^4\right), \tag{1}$$

where A is the black hole's surface area, σ is the Stefan-Boltzmann constant, and T_{BH} and T_{CMB} are the Hawking and CMB temperatures, respectively. A is proportional to the square of the black hole mass [5], T_{BH} is proportional to the inverse of the mass $1/M$ [2], and T_{CMB} depends on the expansion history of the universe. P can be understood as the rate of energy emitted, i.e. $-\mathrm{d}E/\mathrm{d}t$. In $c = 1$ units,

$-\mathrm{d}E/\mathrm{d}t = -\mathrm{d}M/\mathrm{d}t$, so

$$P = -\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{K_{\mathrm{ev}}}{M^2} - M^2 K_{\mathrm{ab}} T_{CMB}^4(t), \tag{2}$$

where K_{ev} is the evaporation constant and K_{ab} is the absorption constant, and $T_{CMB}(t)$ describes the time dependency of the CMB temperature throughout the lifetime of the Universe. It should be noted that we are assuming Hawking Radiation is valid in arbitrarily curved spacetime instead of the weakly curved regime in which it was derived.

With the mass range verified, the next portion of the analysis can begin. In the theory of general relativity, the spacetime geometry is determined by a rank two covariant object called the metric tensor, and is a solution to Einstein's Field Equation s [5]. From this metric tensor, or more commonly referred to as the "metric," a quantity known as the "effective potential" can be derived, which is a function that determines the radial behavior of a particle from the combined effects of the gravitational potential, the centrifugal force, and (in the relativistic limit) the curvature of spacetime. The metric that will be considered is the Reissner-Nordström metric which describes the spacetime around a point source of both mass and electric charge. The effective potential will then be used in to the time-independent Schrödinger Equation , whose solutions are complex-valued quantum mechanical constructs known as "wavefunctions". Instead of describing the exact position and momentum of a quantum particle, these wavefunctions yield the probability of finding a particle in a given volume of phase space. There is no obvious way to merge quantum mechanics and general relativity, and a fully self consistent method of doing so has yet to be found. Our use Schrödinger Equation is at best an approximation to such a consistent theory, and the results must be treated with at least some suspicion. The Schrödinger Equation also does not account for event horizons, which are boundaries in curved spacetime beyond which no events can communicate with events outside of it. The roles of time and space also switch inside these horizons, meaning the metric becomes dynamic instead of static. For simplicity, we shall assume that the wavefunction vanishes at the event horizon and within it, meaning that the probability of finding a particle inside the event horizon vanishes as well. Once the wavefunctions are found, this will allow us to see how the energy levels differ from that of the canonical Hydrogen model when considering the effects of relativity.

The final section of the thesis introduces Numerov's Method [6], an iterative algorithm that can numerically solve certain second order linear differential equations with great accuracy. It also shows how Numerov's Method must be altered if the iteration points are not evenly spaced, which for efficiency purposes they must not be, and the boundary conditions that must be satisfied to find an eigenstate of Schrödinger's Equation .

Located in the Appendix are the computational implementations of several of the equations outlined in the following two sections. They primarily concern solving differential equations numerically in Python, and use both precompiled algorithms in the `scipy` library and an explicit implementation of Numerov's method.

2 Derivation of Black Hole Evaporation Equation

Through analyzing the effects of spacetime curvature on quantum fields, Stephen Hawking discovered that black holes should slowly evaporate and have a temperature proportional to their inverse mass, specifically [7]

$$T_{BH} = \frac{\hbar c^3}{8\pi G k_{\mathrm{B}} M}, \tag{3}$$

where $\hbar$ is the reduced Planck's Constant ($h/2\pi$), c is the speed of light, G is Newton's gravitational constant, k_{B} is Boltzmann's constant, and M is the mass of the black hole. Hawking also showed that the radiation emitted by a black hole should theoretically follow a black body spectrum at a temperature T_{BH}. The spectrum of a black body at a temperature T is given by Planck's Law, which describes the power of radiation emitted per area, per solid angle, per frequency ν, otherwise known as the spectral radiance $I(\nu, T)$. It has the form [8]

$$I(\nu, T) = \frac{2h\nu^3}{c^2 \left(e^{h\nu/k_\mathrm{B}T} - 1\right)}. \tag{4}$$

In order to find the total power P being emitted, this needs to be integrated across all wavelengths and across a full hemisphere (since that is the most that can be seen of a sphere) and multiplied by the surface area of the black body. Radiation pointing directly at an observer also has more of an effect than radiation leaving the black body off the observer-black-body axis by a factor of $\cos\theta$, where θ is the usual spherical angle coordinate measured from that axis. So

$$P(T) = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\infty} I(\nu, T) \cos\theta \sin\theta \, \mathrm{d}\nu \mathrm{d}\theta \mathrm{d}\phi = \sigma A T^4, \tag{5}$$

where $\sigma = \pi^2 k_B^4/(60\hbar^3 c^2)$ is the Stefan-Boltzmann constant [8]. This is the Stefan-Boltzmann Law for power emitted by a black body. Since power is the rate of energy leaving the black hole, this can be rewritten as

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma A T_{BH}^4, \tag{6}$$

where E is the rest energy of the black hole, A is the black hole's surface area, and $T = T_{BH}$. The negative sign is present since the black hole is losing mass, so the rest energy decreases. A black hole's surface area is simply 4π times the square of its radius, which for a non-rotating uncharged black hole is the Schwarzschild radius $r_S = 2GM/c^2$.

The universe itself also has a temperature (which is currently about 2.73 Kelvins) from the cosmic microwave background (CMB), and the radiation associated with that temperature should constantly be bombarding the black hole from all directions and therefore increasing the rest energy of the black hole. This temperature has evolved throughout the universe's history, monotonically decreasing since the Big Bang. Equation 6 can be modified to account for this readily:

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma A[T_{BH}^4 - T_{CMB}^4(t)]. \tag{7}$$

Operating in Planck units where c, G, k_B, $\hbar$, and $4\pi\epsilon_0$ are set to unity, and using the mass-energy relation $E = Mc^2$, we can make Equation 7 dimensionless to find the differential equation governing the mass evolution over time of a black hole:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{1}{15360\pi} M^{-2} + \frac{4\pi^3}{15} M^2 T_{CMB}^4(t), \tag{8}$$

where M, t, and T_{CMB} are all in units of Planck Mass, Planck Time, and Planck Temperature, respectively. These have the following definitions and values in SI units [8]:

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 2.176 \times 10^{-8} \text{ kg}, \tag{9a}$$

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 5.391 \times 10^{-34} \text{ s}, \tag{9b}$$

$$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 1.416 \times 10^{32} \text{ K}. \tag{9c}$$

2.1 Rescaling the Mass Differential Equation

In trying to determine the mass range of primordial black holes that would still exist since they were created at $t \approx 0$, we re-scale time via

$$\begin{aligned} t &\to \frac{t_U}{t_P} t \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} &\to \frac{t_P}{t_U} \frac{\mathrm{d}}{\mathrm{d}t} \end{aligned}$$

where t_U is the age of the universe and t_P is the Planck Time. This rescaling means that $t = 0$ corresponds to the Big Bang and $t = 1$ corresponds to today, making the equation much more manageable when integrated over time. Equation 8 then becomes

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \left[-\frac{1}{15360\pi} M^{-2} + \frac{4\pi^3}{15} M^2 T_{CMB}^4(t) \right] \cdot \frac{t_U}{t_P}, \tag{10}$$

where the ratio t_U/t_P is roughly 8×10^{60}. Being a first-order differential equation, this lends itself nicely to being solved numerically with Runge-Kutta (RK) methods to determine the initial mass range that would still exist since the Big Bang. However we can find a minimum mass analytically by assuming the second term in Equation 10 is negligible throughout the history of the universe. This is to be expected as the Planck Temperature $T_P = \sqrt{\hbar c^5/G k_{\mathrm{B}}^2}$ is roughly 1.417×10^{32} Kelvins, of which the CMB temperature would have been close to for only the first few moments after the Big Bang and quickly drop to well below it. Without this term Equation 10 becomes a first-order separable differential equation, which can be solved by separation of variables.

$$M^2 \,\mathrm{d}M = -\frac{t_U/t_P}{15360\pi} \,\mathrm{d}t. \tag{11}$$

The black hole begins at a mass M_0 at time $t = 0$, and we are trying to find M_0 such that it has zero mass today at $t = 1$. Integrating both sides with these conditions gives

$$\begin{aligned} \int_{M_0}^{0} M^2 \,\mathrm{d}M &= -\frac{t_U/t_P}{15360\pi} \int_0^1 \mathrm{d}t, \\ \frac{1}{3} M_0^3 &= \frac{t_U/t_P}{15360\pi}. \end{aligned}$$

Solving for M_0 gives

$$M_0 = \left(\frac{t_U/t_P}{5120\pi} \right)^{1/3} \approx 7.947 \times 10^{18} \tag{12}$$

in units of Planck masses, which is about $8.7 \times 10^{-20}\ M_{\odot}$. This was calculated using the Planck Collaboration's reported value for their estimation of the age of the universe based on the data taken by the Planck satellite observatory in analyzing the CMB [9].

2.2 Numerical Solutions

Including the second term would require the use of numerical methods like RK algorithms, and would theoretically permit a lower initial mass as the rate of mass loss would be decreased. Python's `scipy` package offers an efficient implementation of an adaptive Runge-Kutta algorithm of order 4 (RK4), meaning that given a step-size h, errors at each step are of order $\mathcal{O}(h^4)$. We are looking for black holes whose initial mass would only allow them to exist up to today, so we will need a root finding algorithm to determine this, which `scipy` also offers through its `brentq` function. Performing this with a CMB temperature of zero (i.e. ignoring the effects of the CMB) corroborates the findings of Equation 12. A graph of the black hole's mass as a function of time for such a choice in temperature appears in Figure

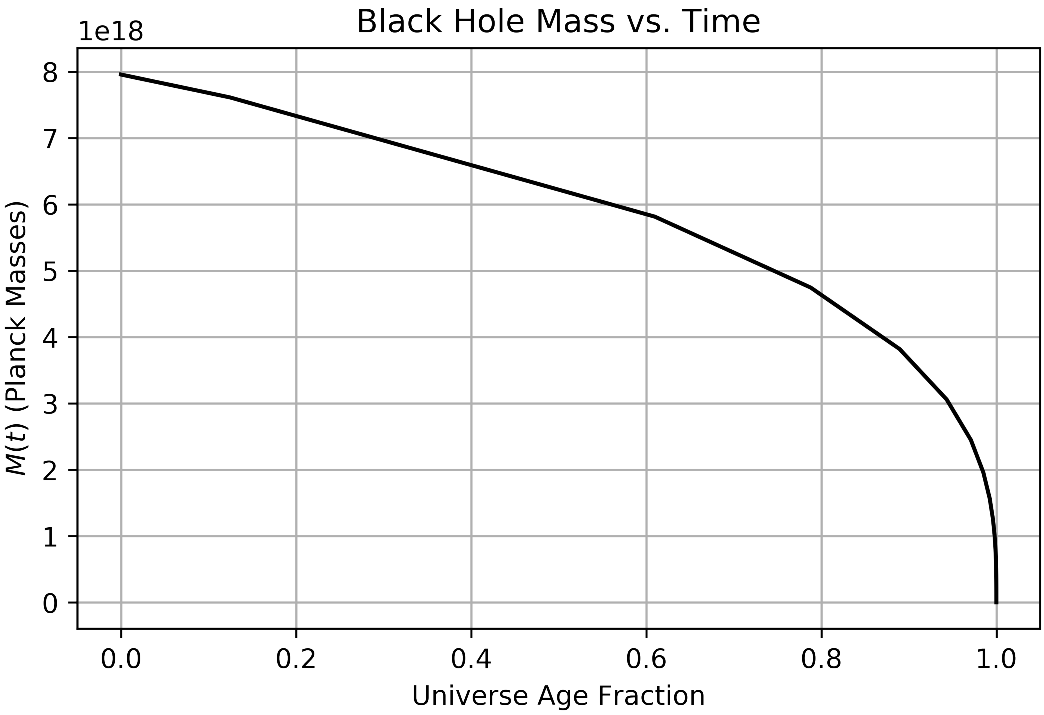

Fig. 1: Black hole mass evolution throughout the history of the universe assuming no effects from CMB radiation.

1. As expected from the form of Equation 10, the mass changes rather slowly initially, but rapidly accelerates to zero as the mass decreases. An implementation of this is given in the Appendix.

2.3 Determining the Temperature Evolution of the CMB

If one would wish to include the effects of the CMB radiation, a well established result of cosmology is that the temperature is proportional to the inverse of the unitless universal scale factor a, meaning the time dependence of a would need to be determined. This depends on the choice of cosmological model, however if the ΛCDM model is chosen, such a time dependence would be the solution to the differential equation [10]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = aH_0\sqrt{\frac{\Omega_{R0}}{a^4} + \frac{\Omega_{M0}}{a^3} + \frac{\Omega_{K0}}{a^2} + \Omega_{\Lambda 0}}, \tag{13}$$

where H_0 is the Hubble Constant and the Ω_0's are the radiation, mass, curvature, and "dark energy" energy densities per the critical energy density (i.e. the density at which the Universe's expansion would continue expanding but eventually stop) at their present values. The most up-to-date estimations for these parameters based on several concordant observations are $\Omega_{R0} \approx 4.6 \times 10^{-5}$, $\Omega_{M0} \approx 0.27$, $\Omega_{\Lambda 0} \approx 0.73$, and Ω_{K0} being negligible [10].

Once the solution to Equation 13 is determined, the temperature of the CMB at any time t is given by [10]

$$T_{CMB}(t) = \frac{T_0}{a(t)}, \tag{14}$$

where T_0 is the temperature of the CMB today, calculated to be about 2.7 Kelvins. This means that in order for there to be any measurable effect in including this term in Equation 10, $a(t)$ must be on the order of 10^{-32} at some time to be comparable to the Plank Temperature. This would only occur very close to the Big Bang. For small a, the a^{-4} term dominates in Equation 13, so it reduces to

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{H_0\sqrt{\Omega_{R0}}}{a}, \tag{15}$$

which has the solution

$$a(t) \propto t^{1/2}. \tag{16}$$

Using the re-scaled time in Equation 10, this would mean that t is on the order of 10^{-64}, which is far beyond the numerical accuracy of 64-bit floating point numbers, and corresponds to around a thousandth of a Planck Time after the Big Bang. Due to the fourth power dependence on temperature in the second term in Equation 10, this term quickly loses any significance over time spans where numerical accuracy is considered, meaning that for our purposes it basically does not matter in determining the minimum mass of black holes that would still exist.

3 The Radial Schrödinger Equation

With the minimum mass required for a black hole created almost contemporaneously with the Big Bang to still exist today determined, we can begin to see how bound quantum states would appear if these black holes managed to capture electrons. It also seems likely that some of these primordial black holes could pick up charge from free protons in the early universe without significantly changing the mass, so that will need to be accounted for as well.

In general, a time independent quantum state $|\psi\rangle$ satisfies the Schrödinger Equation [11]

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \tag{17}$$

where $\hat{H}$ is the Hamiltonian operator, which depends on the dynamics of the particle and the potential it sits in, and E is the energy of the state. This formalism is precisely an eigenvalue problem, where $|\psi\rangle$ is the eigenstate of $\hat{H}$ with eigenvalue E. In three dimensions with a specified coordinate basis, this reads as

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{18}$$

where m is the mass of the particle, ∇^2 is the three dimensional Laplacian operator, and $V(\mathbf{r})$ is the scalar potential. Let us first examine the case of the Hydrogen atom and try to determine the Schrödinger Equation for a bound electron state.

The scalar potential $V(\mathbf{r})$ for the Hydrogen atom is simply the standard spherically symmetric Coulomb potential

$$V(\mathbf{r}) = V(r) = -\frac{e^2}{4\pi\epsilon_0 r}, \tag{19}$$

where e is the electron charge and ϵ_0 is the electric permittivity of free space. Since the potential is spherically symmetric, $\psi(\mathbf{r})$ can be separated into $\psi(\mathbf{r}) = R(r)Y_{lm}(\theta,\phi)$, where Y_{lm} are the spherical harmonic functions [11]. Under this assumption, the dependence on $Y_{lm}(\theta,\phi)$ drops out, leaving only an equation in the radial component of the wave function $R(r)$ and a term representing centrifugal repulsion based on the electron's angular momentum quantum number ℓ. It can be simplified further by defining a function $u(r) = rR(r)$, which leads to the radial Schrödinger equation [11]

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u(r) = Eu(r). \tag{20}$$

The last two terms on the L.H.S. are typically collected into an "effective potential" term $V_{eff}(r)$ where

$$V_{eff}(r) = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}, \tag{21}$$

which describes both the attractive Coulomb force and the repulsive (non-inertial) centrifugal force experienced by the electron. Equation 20 can therefore be rewritten as

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+V_{eff}(r)\right]u(r)=Eu(r). \tag{22}$$

Finding this effective potential for a black hole is key to determining the wavefunction when considering its effects.

3.1 Deriving the Effective Potential

Einstein's Field Theory is a tensor field theory based on the metric tensor, an object that describes curvature of a manifold and how to measure distances on it, and one often looks for solutions to Einstein's Field Equation s that are metric tensors. These objects describe the curvature of four-dimensional spacetime and dictate how matter moves through and affects it. In analyzing the bound states of an electron captured by a charged primordial black hole of the mass discussed in the previous section, the spacetime depends on both the mass of the black hole and its charge. The resulting metric tensor that solves Einstein's Field Equation s is known as the Reissner-Nordström metric, with the line element given by

$$\mathrm{d}s^2=-\left(1-\frac{2GM}{c^2r}+\frac{GQ^2}{4\pi\epsilon_0c^4r^2}\right)c^2\mathrm{d}t^2+\left(1-\frac{2GM}{c^2r}+\frac{GQ^2}{4\pi\epsilon_0c^4r^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2 \tag{23}$$

where M and Q are the mass and charge of the central object, respectfully, and $\mathrm{d}\Omega^2=\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2$ is the spherical solid angle element. Without loss of generality, we can specify that $\theta=\pi/2$ is fixed for any particle orbits, so $\mathrm{d}\Omega^2=\mathrm{d}\phi^2$. For simplicity, we will rewrite Equation 23 as

$$\mathrm{d}s^2=-c^2\mathrm{d}\tau^2=-fc^2\mathrm{d}t^2+f^{-1}\mathrm{d}r^2+r^2\mathrm{d}\phi^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{24}$$

from now on, where $f=\left(1-\frac{2CM}{c^2r}+\frac{GQ^2}{4\pi\epsilon_0c^4r^2}\right)$ and τ is the proper time (time the particle measures in its own frame). From the principle of least action, a test particle will follow the path in spacetime that minimizes a quantity known as the "action", which is the integral of the Lagrangian $\mathcal{L}$ over the proper time experienced. In order to minimize the action, $\mathcal{L}$ must satisfy the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial x^\mu}=\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu},\quad\text{where}\quad\dot{x}^\mu\equiv\frac{\partial x^\mu}{\partial\tau}, \tag{25}$$

which represents four equations for $\mu=0$, 1, 2, and 3. Integral to this analysis are conserved quantities throughout the particle's motion. If $\mathcal{L}$ does not explicitly depend on a coordinate x^μ (i.e. $\partial\mathcal{L}/\partial x^\mu=0$), then a conserved quantity exists related to that coordinate since Equation 25 would imply $\partial\mathcal{L}/\partial\dot{x}^\mu$ is constant throughout the particle's trajectory. We will define the Lagrangian to be [12]

$$\mathcal{L}=\frac{1}{2}mg_{\mu\nu}\dot{x}^\mu\dot{x}^\nu+qA_\mu\dot{x}^\mu=\frac{1}{2}m(-fc^2\dot{t}^2+f^{-1}\dot{r}^2+r^2\dot{\phi}^2)+qA_\mu\dot{x}^\mu, \tag{26}$$

where m and q are the mass and charge of the particle, respectfully, $A_\mu=(\Phi,A_r,A_\theta,A_\phi)$ is the electromagnetic four-potential with $\Phi=Q/4\pi\epsilon_0r$ being the Coulomb potential. A_r, A_θ, and A_ϕ are the covariant components of the vector potential, but we can operate in a gauge where these are zero since there are no moving charges or current densities [13]. For this analysis we will take the quantity $A_\mu\dot{x}^\mu$ to be

$$A_\mu\dot{x}^\mu=\eta_{\mu\nu}A^\mu\dot{x}^\nu=-\frac{qQ}{4\pi\epsilon_0r}\dot{t} \tag{27}$$

where we have used the flat-spacetime Minkowski metric tensor since electromagnetic interactions should not be affected by spacetime curvature.

With $\mathcal{L}$ written out in full, it becomes apparent from Equation 25 what the conserved quantities are throughout the particle's path since $\mathcal{L}$ does not explicitly depend on t or ϕ. The Euler-Lagrange equations tell us that $\partial\mathcal{L}/\partial\dot{t}$ and $\partial\mathcal{L}/\partial\dot{\phi}$ are conserved quantities which we will define as

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = -mc^2 f\dot{t} - \frac{qQ}{4\pi\epsilon_0 r} \equiv -E, \tag{28}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi} \equiv L, \tag{29}$$

where E and L are the energy and orbital angular momentum of the particle, respectfully. Solving for $\dot{t}$ and $\dot{\phi}$ yields:

$$\dot{t} = \frac{1}{mc^2 f}\left(E - \frac{qQ}{4\pi\epsilon_0 r}\right), \tag{30}$$

$$\dot{\phi} = \frac{L}{mr^2}, \tag{31}$$

Dividing Equation 24 by $\mathrm{d}\tau^2$ is the final step to this process. For massive particles, the fraction $\mathrm{d}\tau/\mathrm{d}\tau = 1$ since they experience time, as opposed to massless particles like photons who do not. Upon plugging in the expressions for $\dot{t}$ and $\dot{\phi}$, this yields

$$c^2\left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau}\right)^2 = c^2 = c^2 f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\phi}^2, \tag{32a}$$

$$= c^2 f\frac{1}{f^2 m^2 c^4}\left(E - \frac{qQ}{4\pi\epsilon_0 r}\right)^2 - f^{-1}\dot{r}^2 - r^2\frac{L^2}{m^2 r^4}, \tag{32b}$$

$$= \frac{1}{fm^2c^2}\left(E - \frac{qQ}{4\pi\epsilon_0 r}\right)^2 - f^{-1}\dot{r}^2 - \frac{L^2}{m^2 r^2}, \tag{32c}$$

Some rearranging of terms gives us an equation for the total energy E:

$$E = \frac{qQ}{4\pi\epsilon_0 r} + mc^2\sqrt{f\left(1 + \frac{L^2}{m^2c^2r^2}\right) + \frac{\dot{r}^2}{c^2}}, \tag{33}$$

where we have chose the positive root since the negative root would correspond to a particle moving backward in time. If we relate this equation to a classical particle under the influence of physical potentials, we can determine the effective potential by setting $\dot{r} = 0$, i.e. the energy the particle has at its turning points. So

$$V_{eff}(r) = \frac{qQ}{4\pi\epsilon_0 r} + mc^2\left[\sqrt{\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r^2}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)} - 1\right], \tag{34}$$

where the rest energy of the particle has been subtracted off to have the potential go to zero at infinity. This will be the effective potential used in Equation 22 to determine the wavefunction.

3.2 Working Without Units

With the effective potential around a charged black hole determined in Equation 34 and plugged into Equation 22, we can begin altering it to make it dimensionless. The constants present in Equation 22 span many orders of magnitude in SI units which makes it not optimized for computational methods, and the computer will have a much easier time solving this equation if they are not present.

In order to accomplish this, we will follow the standard approach of setting each variable quantity equal to a constant of the same units multiplied by a unitless variable. The only variables in Equation 22 are the distance from the origin r, the angular momentum L, and the energy E. We will make the following substitutions:

$$r = a_0\rho = \frac{4\pi\epsilon_0\hbar^2}{me^2}\rho, \tag{35}$$

$$L^2 = \ell(\ell+1)\hbar^2, \tag{36}$$

$$E = \mathrm{Ry}\varepsilon = \frac{1}{2}\frac{e^4 m}{16\pi^2\epsilon_0^2\hbar^2}\varepsilon, \tag{37}$$

where a_0 is the Bohr radius ($\approx 5.292 \times 10^{-11}$ meters), m and e are the mass and charge of the electron, respectfully, and Ry is the Rydberg Constant ($\approx$ 13.6 eV). We have also mimicked the quantum mechanical effect of the way the square of angular momentum scales with the angular momentum quantum number ℓ. This rescaling of r requires us to redefine the differential operator, which is accomplished through the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}\rho}, \tag{38a}$$

$$= \frac{1}{a_0}\frac{\mathrm{d}}{\mathrm{d}\rho}, \tag{38b}$$

$$\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}r^2} = \frac{1}{a_0^2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} = \frac{m^2e^4}{16\pi^2\epsilon_0^2\hbar^4}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}. \tag{38c}$$

Plugging in Eqs. 35 and 36 into the relativistic effective potential Equation 34 we get

$$\begin{aligned} V_{eff}(\rho) = &-\frac{me^4}{(4\pi\epsilon_0\hbar)^2}\frac{1}{\rho} + \\ &mc^2\left[\sqrt{\left(1 - \frac{2GMme^2}{4\pi\epsilon_0(\hbar c)^2}\frac{1}{\rho} + \frac{Gm^2e^6}{(4\pi\epsilon_0)^3(\hbar c)^2}\frac{1}{\rho^2}\right)\left(1 + \frac{e^4}{(4\pi\epsilon_0\hbar c)^2}\frac{\ell(\ell+1)}{\rho^2}\right)} - 1\right], \end{aligned} \tag{39}$$

which does not seem to have made anything easier. However, it turns out that most of these terms are powers of the fine structure constant α, defined as

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}. \tag{40}$$

Using this fact, $V_{eff}(\rho)$ becomes

$$\frac{V_{eff}(\rho)}{mc^2} = -\frac{\alpha^2}{\rho} + \sqrt{\left(1 - \frac{2GMm}{\hbar c}\frac{\alpha}{\rho} + \frac{Gm^2}{\hbar c}\frac{\alpha^3}{\rho^2}\right)\left(1 + \alpha^2\frac{\ell(\ell+1)}{\rho^2}\right)} - 1 \tag{41}$$

which is much simpler, but still quite unwieldy. We can make things a bit easier by realizing that the quantity attached to the α^3/ρ^2 term in the square root is quite small

$$\frac{Gm^2\alpha^3}{\hbar c} \approx 6.81 \times 10^{-52},$$

which behooves us to expand this potential about orders of $1/\rho$ since the contributions must drop off fairly quickly. Using the computer algebra software Wolfram|Alpha, $V_{eff}(\rho)$ can be expressed up to second order in $1/\rho$ as

$$\frac{V_{eff}(\rho)}{mc^2} = -\left[\alpha^2 + \frac{GMm\alpha}{\hbar c}\right]\frac{1}{\rho} + \frac{1}{2}\left[\left(\frac{GMm\alpha}{\hbar c}\right)^2 + \alpha^2\ell(\ell+1)\right]\frac{1}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right), \tag{42}$$

where the contribution from the $Gm^2\alpha^3/\hbar c$ term has been ignored. We will only be considering terms up to and including $1/\rho^2$ since beyond that the limits of floating point numbers begins to become a significant factor. Plugging this approximation into Equation 22 along with the rescaled variables Eqs. 35-37 and the definition for α, we arrive at

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{2}{\alpha^2}\left(-\left[\alpha^2 + \frac{GMm\alpha}{\hbar c}\right]\frac{1}{\rho} + \frac{1}{2}\left[\left(\frac{GMm\alpha}{\hbar c}\right)^2 + \alpha^2\ell(\ell+1)\right]\frac{1}{\rho^2}\right)\right]u(\rho) = -\varepsilon u(\rho). \quad (43)$$

We can see terms that the terms that normally appear in the effective potential have been modified slightly and in fact go to zero when the mass of the central object goes to zero, making these the relativistic corrections up to second order in $1/\rho$. After collecting terms we get

$$\frac{\mathrm{d}^2 u(\rho)}{\mathrm{d}\rho^2} = \left[-\frac{2}{\rho} + \frac{\ell(\ell+1)}{\rho^2} - \frac{2GMm}{\alpha\hbar c}\frac{1}{\rho} + \left(\frac{GMm}{\hbar c}\right)^2\frac{1}{\rho^2} - \varepsilon\right]u(\rho), \quad (44)$$

which is a fully unitless differential equation. Before integrating this to find the wavefunctions, the aforementioned condition at the event horizon of this black hole must be considered. For a Reissner-Nordström black hole, there exists two event horizons $r_\pm$ located at

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\frac{G^2M^2}{c^4} - \frac{GQ^2}{4\pi\epsilon_0 c^4}}, \quad (45)$$

which are the roots of the factor f in Equation 24. For a black hole of 10^{19} Planck masses and the charge of a proton, the outer event horizon is about 6.1×10^{-6} Bohr radii. This is in fact less than the radius of the proton which, in determining the electron orbitals for the non-relativistic case, is typically ignored. Following in that vein, the event horizon will be ignored as well.

4 Numerov's Method

The form of both the standard radial Schrödinger Equation (Equation 20) and the one including relativistic perturbations (Equation 44) lend themselves nicely to utilizing Numerov's method to numerically determine their solutions. Compared to other numerical methods, such as Runge-Kutta, it is more accurate but more computationally expensive, meaning that some optimizations need to be made to make it viable for this research. To investigate this, let's examine Equation 20 after performing the same rescaling process we did for Equation 44 as an example:

$$\frac{\mathrm{d}^2 u(\rho)}{\mathrm{d}\rho^2} = \left[\frac{\ell(\ell+1)}{\rho^2} - \frac{2}{\rho} - \varepsilon\right]u(\rho). \quad (46)$$

This has the form required to use Numerov's method, a numerical technique that solves differential equations of the family

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} = g(x)y(x). \quad (47)$$

Given a set of integration points x_i evenly separated by a distance h, the solution can be found recursively via [6]:

$$y_{i+1}\left(1 - \frac{h^2}{12}g_{i+1}\right) = 2y_i\left(1 + \frac{5h^2}{12}g_i\right) - y_{i-1}\left(1 - \frac{h^2}{12}g_{i-1}\right) + \mathcal{O}(h^6), \quad (48)$$

where $y_i = y(x_i)$ and $g_i = g(x_i)$. This is Numerov's method, which has $\mathcal{O}(h^6)$ accuracy. Since the original ODE is second order, we need two points for the solution to begin the iteration, hence the y_i

and y_{i-1} terms. For $\rho \gg 1$, Equation 46 reduces to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} \approx -\varepsilon u, \tag{49}$$

which has solutions of the form

$$u(\rho \gg 1) = A \exp(-\sqrt{\varepsilon}\rho) + B \exp(\sqrt{\varepsilon}\rho). \tag{50}$$

To enforce normalizability of the wavefunction, we set $B = 0$ so the solution decays as ρ increases. This exponential decay behavior means that for large ρ the solution should not look very dramatic, meaning that we would be wasting computational power by treating this region of integration as precisely as we would the small ρ region. To account for this, we will choose a logarithmically spaced grid of integration points ρ_i defined by

$$\rho_i = \exp(x_i), \tag{51}$$

where

$$x_{i+1} = x_i + h \quad ; \quad x_i \in (-\infty, \infty), \tag{52}$$

with h being the spacing between adjacent x_i. Performing this change of variables assigns less computational weight to larger r while maintaining the accuracy necessary towards the center. Now we need to calculate the first derivative operator under this new coordinate definition, which follows directly from the chain rule of differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}\rho} = \frac{\mathrm{d}x}{\mathrm{d}\rho}\frac{\mathrm{d}}{\mathrm{d}x}, \tag{53}$$

$$= \left(\frac{\mathrm{d}\rho}{\mathrm{d}x}\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}x}, \tag{54}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}r} = \exp(-x)\frac{\mathrm{d}}{\mathrm{d}x}. \tag{55}$$

Doing this again for the second derivative operator:

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} = \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\exp(-x)\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{56}$$

$$= \exp(-x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp(-x)\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{57}$$

$$\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} = \exp(-2x)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{58}$$

This presents a problem for numerically determining the wavefunction with the Numerov method, as we have now introduced a first derivative term. We therefore need to define $u(\rho)$ in such a way as to remove it. This can be accomplished through the transformation [14]

$$u(\rho) = \exp(x/2)F(x). \tag{59}$$

Finding $\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2}$ after the change of variables from Equation 58 yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} = \exp(-3x/2)\left(\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - \frac{F}{4}\right). \tag{60}$$

Plugging this and Equation 59 back into Equation 47:

$$e^{-3x/2}\left(\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - \frac{F}{4}\right) = g(e^x)\cdot e^{x/2}F, \tag{61}$$

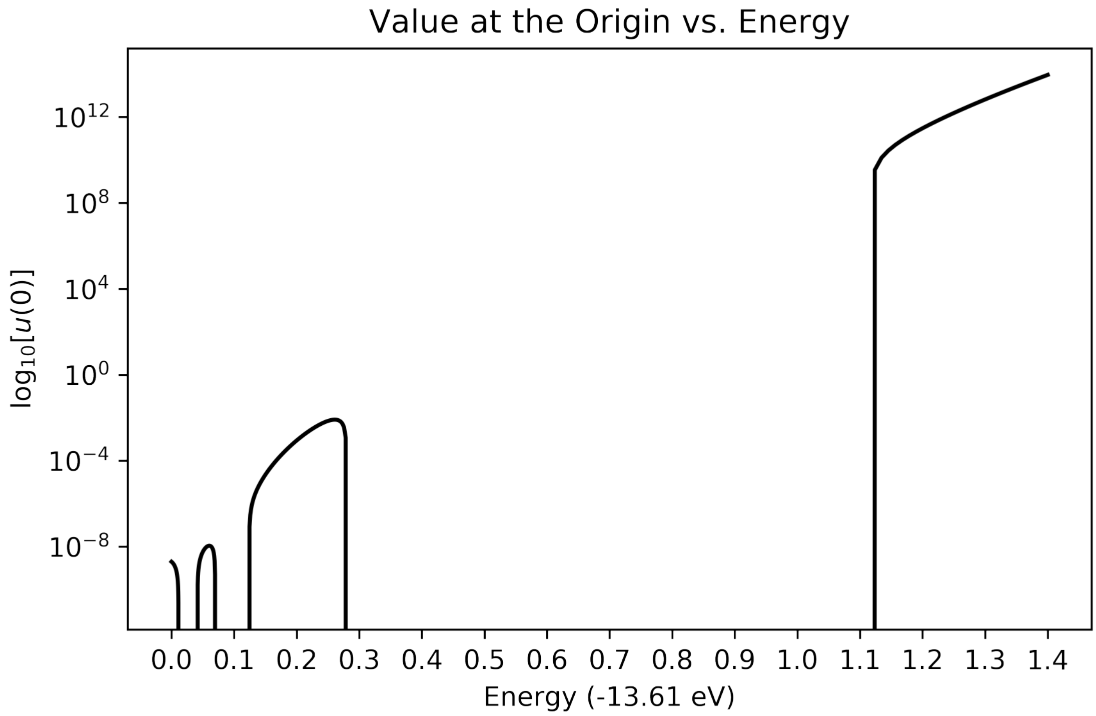

Fig. 2: Plot of the value of the solution at the origin for a given energy. Where the plot goes below the E axis is where it is negative.

$$\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}x^2}F(x) = \tilde{g}(x)F(x), \tag{62}$$

where

$$\tilde{g}(x) = e^{2x}g(e^x) + \frac{1}{4}. \tag{63}$$

This equation now has the form required for using Numerov's method, with the solution we seek obtained by using Equation 59.

4.1 Performing the Numerical Integration

With this logarithmically spaced grid used to achieve high accuracy while reducing computation time, we can now use Equation 48 to find solutions to Equation 62. In the case of considering the relativistic effects, the function $g(\rho)$ in Equation 63 is

$$g(\rho) = -\frac{2}{\rho} + \frac{\ell(\ell+1)}{\rho^2} - \frac{2GMm}{\alpha\hbar c}\frac{1}{\rho} + \left(\frac{GMm}{\hbar c}\right)^2\frac{1}{\rho^2} - \varepsilon, \tag{64}$$

which is the same as the function preceding $u(\rho)$ on the right hand side of Equation 44. In order to implement this into Python code, we need initial conditions. Integrating from the outside in is preferable to integrating from the origin since it avoids the issues associated to being near a singularity. Sufficiently far away from the origin the solution can be expanded in a Taylor series:

$$F(x-h) = F(x) - h\frac{\mathrm{d}F}{\mathrm{d}x} + h^2\frac{1}{2}\frac{\mathrm{d}^2F}{\mathrm{d}x^2} + \dots . \tag{65}$$

The point $F(x)$ will be set to zero, and the next point will be calculated using a small negative value for $\mathrm{d}F/\mathrm{d}x$ (such as -10^{-8}) and multiplied by the step size, with higher order terms being ignored. This serves to ensure the function is integrable by approximating the vanishing at infinity condition, and also

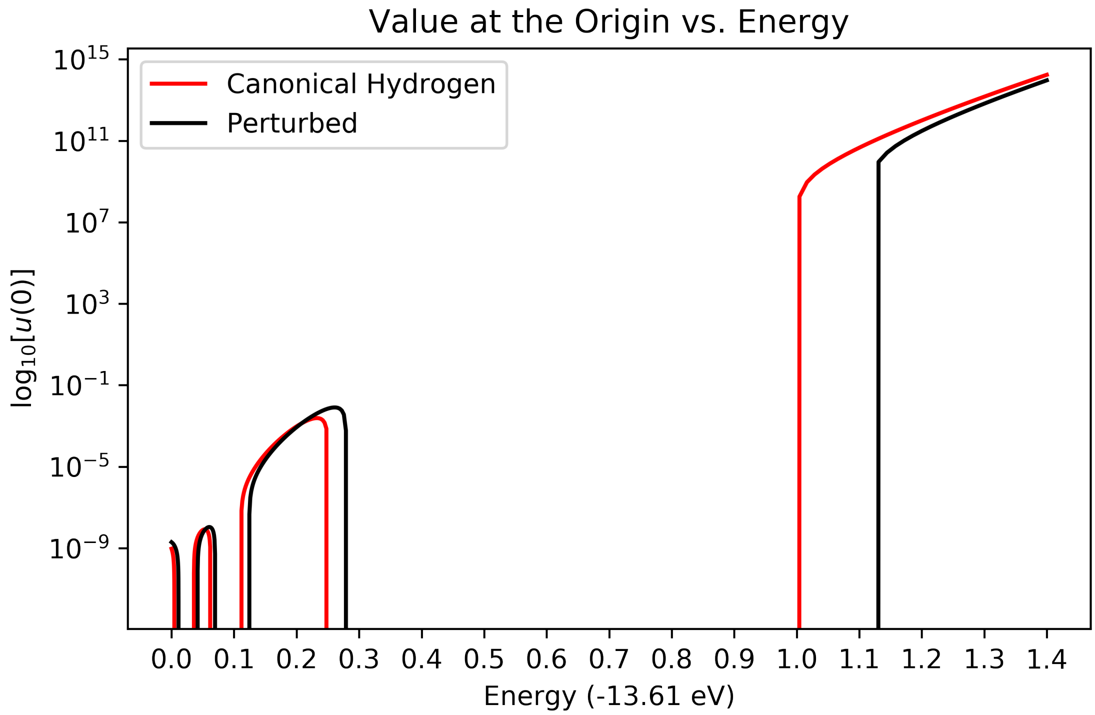

Fig. 3: Overlay of the canonical Hydrogen energy levels and the levels associated with the perturbed Hydrogen potential.

serves as the two points required to initiate Numerov's method.

Another consideration that needs to addressed is the boundary condition at $r = 0$. From our choice of $R(r) = u(r)/r$, u must go to zero at least as fast as linearly, for if it is some non-zero value, $R(r)$ will diverge. To accomplish this, we will employ the "shooting method" through varying ε in Equation 64 until $u(r = 0) \approx 0$, which will be done through a similar root finding method in Section 2.2.

The value for M in Equation 64 will be approximately the value obtained in Section 2.2 of 10^{19} Planck masses, and m will be the electron mass. Equation 64 then becomes

$$g(\rho) = -\frac{2}{\rho} + \frac{\ell(\ell+1)}{\rho^2} - \frac{0.1147}{\rho} + \frac{1.752 \times 10^{-7}}{\rho^2} - \varepsilon. \tag{66}$$

Our implementation in Python code is presented in the Appendix. From Fig. 2, the first root occurs somewhere between -1.1 and -1.2 Ry, with the next roots coming in successively closer intervals. This follows the same pattern as the canonical Hydrogen case where the energy levels appear at -Ry$/n^2$; $n \in \mathbb{N}$, overlaid with the relativistic effective potential result in Fig. 3. This shows us that the energy levels are slightly altered from the Hydrogen model when considering the effects of general relativity, specifically that the electron is slightly more bound. This effect is quite marginal, however, and requires an extreme central mass to be noticeable.

5 Conclusion

Under the assumption that Hawking Radiation is valid, a black hole should evaporate at a rate proportional to the inverse square of its mass if left alone, differing from the eternal black hole solution that General Relativity suggests. Taking into account the cosmic microwave background radiation bombarding the black hole throughout the entire history of the universe, which would serve to increase the mass over time, we get a full picture of the mass history of a black hole created at the Big Bang. However, the contribution from the CMB is basically negligible, so the evaporation rate of a black hole is almost entirely determined by Hawking Radiation. Solving the resulting mass differential equation with the appropriate integration bounds, the lowest initial mass of black hole created at the Big Bang that would completely evaporate today is about 7.95×10^{18} Planck masses, or about 1.7×10^{11} kilograms. This is in agreement with previous research proposing that these primordial black holes could be a candidate particle for dark matter.

Using the metric for a Reissner-Nordström black hole (i.e. a black hole that contains both mass and charge, but no angular momentum), the effective radial potential was derived. This was then used in the time independent radial Schrödinger Equation for a spherically symmetric potential. This is not self-consistent as there is no known way to properly merge quantum mechanics and general relativity, however this is at best an approximation. Using the mass obtained in the previous section, we find that an electron forms a more tightly bound state to a source of that mass and the charge of a proton than to a proton alone. This system would still create absorption and emission lines like a normal Hydrogen atom, however the slightly different energy levels would mean the spectrum would appear Hydrogen-like but different, meaning such spectra from these black hole bound states could be observable. It is clear from Equation 66 that the including the effect of the black hole primarily alters the $1/\rho$ and $1/\rho^2$ terms. Judging by their respective signs, the charge is increased and there exists a small repulsive centrifugal-like term even if the electron's angular momentum is zero.

Numerov's method is a very accurate way of numerically integrating certain types of second-order linear differential equations which often show up in physical systems. However, this accuracy does not permit much computational efficiency, compounded by the fact that it requires a uniformly spaced grid of integration points. In order to get around this, we took advantage of the shape of solutions to the radial Schrödinger Equation near the origin, and discovered that most of the features occur in that region. It then makes sense to treat these points with more precision that points far away from the origin. This was accomplished by an exponential change of variables $\rho = \exp(x)$, where $x \in \mathbb{Z}$ as opposed to the radial coordinate $\rho \in \mathbb{Z}^+$. This mapping can take a uniformly spaced x grid and transform it into a logarithmically spaced grid in ρ, where the spacing between points gets smaller as one approaches the origin. Using the proper differential operator in this new space and an appropriate redefinition of the solution function, we arrived at a differential equation in the form required by Numerov's method. Since the desired high accuracy near the origin is now possible with far fewer integration points, this greatly improves the efficiency of solving the differential equation.

6 Appendix

6.1 Mass Evolution of a Black Hole

Presented below is a Python implementation of numerically solving Equation 10. The main features are the `scipy` functions `solve_ivp` and `brentq`, which are an RK4 algorithm and a root finding algorithm, respectively. An arbitrary temperature function `T(t)` can be used, however Fig. 1 was created using `T(t) = 0`.

```python
import numpy as np # Used to define constants
from scipy.integrate import solve_ivp # ODE solver
from scipy.optimize import brentq # Root finder

# Creating evaporation and absorption constants
Kev = 1/(15360*np.pi)
Kab = 4*(np.pi**3)/15

# Defining equation to be integrated w/ some
# temperature function T(t)
def dMdt(t, M):
```

```
return (-Kev/(M**2) + Kab*(M**2)*(T(t)**4))*8e60

# Time span the solver integrates over
t_span = [0, 1.5]

# Defining a function that takes in an initial mass, solves the
# ODE, and returns the last evaluated time of the solver minus 1.
# Therefore an initial mass that lives only to today would return
# a value of zero.
def minMass(M0):
# Solver ends automatically when M reaches 0 as dMdt becomes undefined
result = solve_ivp(dMdt, t_span, [M0])
# Returning the last evaluated time minus 1
return result.t[-1] - 1

# Finding the root of minMass between minMass(1e18) and minMass(1e19)
mass = brentq(minMass, 1e18, 1e19)
```

6.2 Numerov's Method in Python

Presented below is a step-by-step process for implementing the solving of the "relativistic" Schrödinger's Equation on the logarithmic grid, Equation 62. First we will import useful libraries and define Eqs. 64 and 63:

```
import numpy as np # Various functions and array manipulation
from scipy.optimize import brentq # Root finding function

# Defining the variable coefficient in the radial Schroedinger equation
def g(r, l, E):
return -2/r + l*(l+1)/(r*r) - .1147/r + 1.752e-7/(r*r) - E
# Variable coefficient after coordinate transformation and function redefinition
def gTilda(x, l, E):
return np.exp(2*x)*g(np.exp(x), l, E) + 0.25
```

Next, the evaluation points and an empty array of the same length is created, the latter of which will have its elements be iteratively assigned following Equation 48. Also defined are the quantum numbers ℓ, k, and n, the latter two corresponding to the number of nodes in the solution and the energy level in the canonical Hydrogen case, respectively. The order of the evaluation points are also reversed so the integration is performed backwards.

```
l, k = 0, 0 # angular momentum and number of nodes
n = 1 + l + k # energy level

h = 1e-4 # step size
# Creating array of evaluation points and reversing their order
x = np.arange(np.log10(1e-6), np.log10(100), h)[::-1]
r = np.exp(x)

# Creating the empty array that will eventually hold the solution
F = np.empty(len(x))
```

```
F[0], F[1] = 0, 1e-8*h # Initial conditions

# Defining the function that calculates the result given the above
# definitions, an energy E, and angular momentum l
def Numerov(E, l, F, x, h):
for i in range(x.shape[0] - 2):
F[i+2] = (2*(12 + 5*h*h*gTilda(x[i+1], l, E))*F[i+1] -
(12 - h*h*gTildaP(x[i], l, E))*F[i])/(12 -
h*h*gTildaP(x[i+2], l, E))
return F
```

Now we will define a function that linearly approximates the value of the result at $r = 0$ as a function of the energy E. What this function returns is then passed into the brentq function so the value of E can be found which makes the result vanish at $r = 0$.

```
def root(E):
# Transforming the solution to r space
u = Numerov(E, l, F, x, h)*np.exp(x/2)
# Linearly approximating the result to the origin
return u[-1] + ((u[-2]-u[-1])/(r[-2]-r[-1]))*(0.0-r[-1])
```

The function brentq needs two evaluation points for the function it is finding the roots of, and the signs of the evaluations must be oppositely signed. We can plot root(E) for various values of E to visually determine the ranges to search, as shown in Figure 2.

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