# Dirac Delta Function

first $3\frac{1}{2}$ pages



From Gauss's law we know that

$$\int_V d\tau\, \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = -\int_V d\tau\, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = \begin{cases} -4\pi \\ 0 \end{cases}$$

depending on whether the integration includes the origin $\vec{r}=0$ or not. This result is conveniently expressed by introducing the Dirac delta function,

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r}) = -4\pi\,\delta(x)\,\delta(y)\,\delta(z).$$

This Dirac delta function is defined by the following assigned properties

$$\delta(x) = 0 \quad \text{if} \quad x \neq 0$$

$$f(0) = \int_{-\infty}^{\infty} dx\, f(x)\,\delta(x) \tag{1}$$

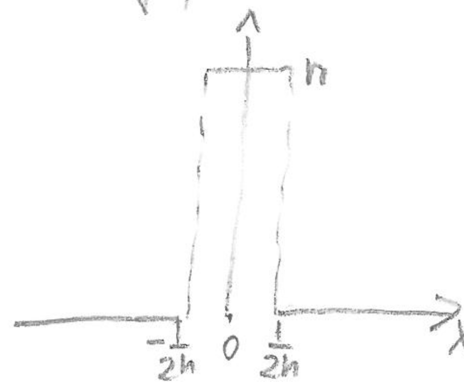
where $f(x)$ is a well-behaved function and the integration includes the origin. As a special case we have $(f(x)=1)$

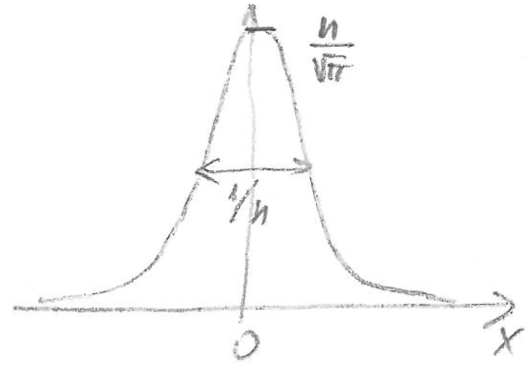
$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1 .$$

For this to be satisfied $\delta(x)$ must be an infinitely high spike at $x=0$. This physically describes an impulsive force or the charge density for a point charge. However, no such function exists in the usual sense of function. The delta function has to be defined as a <u>distribution</u>.

The crucial property (1) can be developed as the limit of a sequence of functions. Examples of such sequences of functions in $n$ for $n \to \infty$ are the following

$$* \quad \delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$$



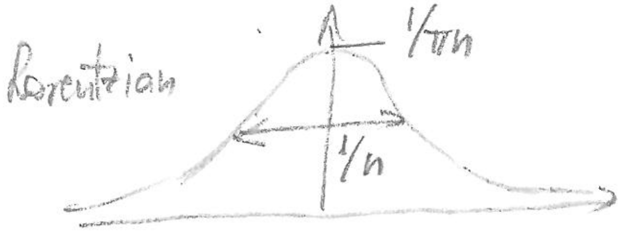



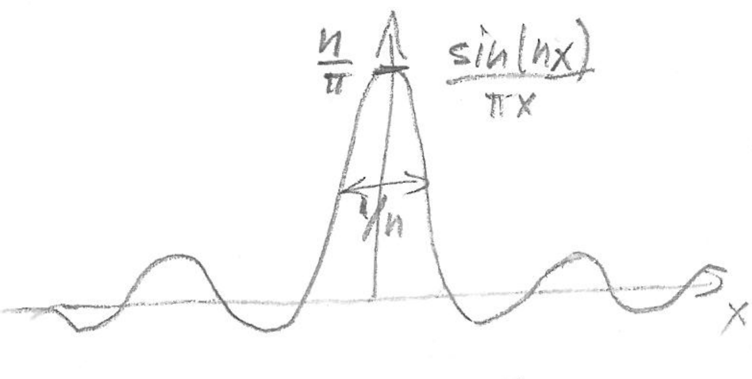

$$* \quad \delta_n(x) = \frac{n}{\sqrt{\pi}} \exp(-n^2 x^2)$$

normalized Gaussian

$$* \quad \delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2} = \frac{1}{\pi} \frac{1/n}{x^2 + 1/n^2}$$ Lorentzian of width $1/n$

$$* \quad \delta_n(x) = \frac{\sin(nx)}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} dt\, e^{ixt}$$




We now check the normalization of these functions: $\int_{-\infty}^{\infty} dx\, \delta_n(x) = 1$

$$* \quad \int_{-\infty}^{\infty} dx\, \delta_n(x) = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} dx\, n = n\left(\frac{1}{2n} - \left(-\frac{1}{2n}\right)\right) = 1 \quad \checkmark$$

* Gaussian $\displaystyle\int_{-\infty}^{\infty} dx \frac{n}{\sqrt{\pi}} \exp(-n^2x^2) = \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{\pi}} e^{-y^2} \qquad y = nx$

Square the integral and use polar coordinates

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dy\, e^{-y^2} \int_{-\infty}^{\infty} dz\, e^{-z^2} = \frac{1}{\pi}\int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\theta\, e^{-\rho^2} = 2\,\frac{1}{2}\int_0^{\infty} d\rho^2\, e^{-\rho^2} = 1 \quad \checkmark$$

* Lorentzian $\displaystyle\int_{-\infty}^{\infty} dx \frac{n}{\pi} \frac{1}{1+nx^2} = \int_{-\infty}^{\infty} dy \frac{1}{\pi} \frac{1}{y^2+1} \qquad y = nx$

$$= \frac{1}{\pi} \arctan(y) \Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1 \quad \checkmark$$

* For $\displaystyle \delta_n(x) = \frac{\sin(nx)}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} dt\, e^{ixt}$

<u>Dirichlet kernel</u>

we may approximate

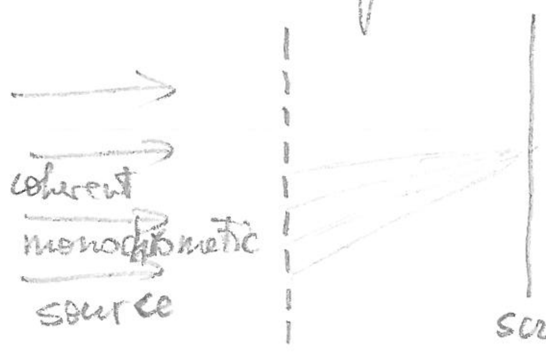
$$\delta_n(x) = \frac{1}{2\pi} \frac{\sin[(n+\frac{1}{2})x]}{\sin(\frac{x}{2})} = \frac{1}{2\pi} \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{\frac{ix}{2}} - e^{-ix/2}}$$

$$= \frac{1}{2\pi} e^{inx} \frac{1 - e^{-i(2n+1)x}}{1 - e^{-ix}} = \frac{1}{2\pi} e^{inx} \sum_{l=0}^{2n} e^{-ilx} =$$

$$= \frac{1}{2\pi} \sum_{l=0}^{2n} e^{i(n-l)x} = \frac{1}{2\pi} \sum_{l=-n}^{n} e^{ilx}$$

This sum appears in physics as the sum over plane waves in a diffraction pattern.

<u>optical path differences</u>
for light or in multislit interference in QM.



For the normalization of the Dirichlet kernel, we look into an integration table:

$$\int_0^\infty dx \, \frac{\sin(ax)}{x} = \frac{\pi}{2} \, \mathrm{sign}(a) \, . \qquad \text{(independent of the magnitude of } a\text{)}$$

To prove this it is convenient to use calculus of a complex variable, in particular Cauchy's theorem and a contour integral. Hence, we have to skip the proof at this point.

Note that from a mathematical standpoint the limit

$$\lim_{n\to\infty} \delta_n(x) \qquad \text{does not exist}$$

end of class

What has an existing limit is the integral over the kernel:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) \, f(x) \, dx = f(0) \qquad \text{since} \quad \int_{-\infty}^{\infty} dx \, \delta_n(x) = 1$$

for all $n$. We may denote this (leaving out the $\lim_{n\to\infty}$) in a formal way as

$$\int_{-\infty}^{\infty} dx \, \delta(x) \, f(x) = f(0) \, ,$$

but the limiting process of the integral is implicitly assumed. From the definitions of the kernels we can infer that the $\delta$-function distribution is even in its argument: $\delta(-x) = \delta(x)$.

The delta function has other important properties:

* $\delta(ax) = \frac{1}{|a|} \delta(x)$, which follows from the normalization from rescaling the differential $dx$.

* assume that $g(x)$ is a function with <u>simple zeros</u>, i.e. it has no double root in the Taylor expansion about a zero, then

$$\delta[g(x)] = \sum_{\substack{a \\ g(a)=0 \\ g'(a)\neq 0}} \frac{\delta(x-a)}{|g'(a)|}$$

The proofs of these properties are

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(ax) = \frac{1}{|a|}\int_{-\infty}^{\infty} dy\, f\!\left(\tfrac{y}{a}\right) \delta(y) = \frac{1}{|a|} f(0) \qquad y = |a|x$$

and expanding in a Taylor series about each simple zero of $\delta(x)$

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta[g(x)] = \sum_a \int_{a-\varepsilon}^{a+\varepsilon} dx\, f(x)\, \delta[(x-a)g'(a)] = \sum_a \frac{1}{|g'(a)|} f(a)$$

where we kept only the first term of the Taylor expansion and used the previous property. The integration limit can be extended to $\infty$ because the $\delta$-function is zero unless $x=a$.

We now turn to a few examples:

* Total charge inside a sphere: The total electric flux $\oint_S \vec{E}\cdot d\vec{\sigma}$ out of a sphere of radius $R$ about the origin is $\frac{\sum_j e_j}{\varepsilon_0}$, where we sum over the charges $e_j$ located inside the sphere, i.e. at positions $|\vec{r}_j| < R$.

We know that $\vec{E} = -\vec{\nabla}\varphi(\vec{r})$, where $\varphi$ is the potential

$$\varphi(\vec{r}) = \sum_{j=1}^{n} \frac{e_j}{|\vec{r}-\vec{r}_j|} = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} .$$

Here we assumed $n$ charges inside the sphere. The total charge density is $$\rho(\vec{r}) = \sum_j{}' e_j\, \delta(\vec{r}-\vec{r}_j)$$

where the charge $e_j$ is the intensity of the $\delta$-function at $\vec{r}_j$. Note that $\rho(\vec{r})$ is highly singular. It is zero everywhere except at the points where the charge is located, where $\rho(\vec{r})$ has a spike. The limiting process of the delta kernel smears the portion of the charge. We have now

$$\oint_S \vec{E}\cdot d\vec{\sigma} = -\oint_S \vec{\nabla}\varphi \cdot d\vec{\sigma} = -\int_V d\vec{r}^3\, \nabla^2\varphi = \int_V d\vec{r}^3\, \frac{\rho(\vec{r})}{\varepsilon_0} = \frac{1}{\varepsilon_0}\sum_j{}' e_j\,,$$

where we used Gauss's theorem and $\int d\vec{r}^3 \rho(\vec{r}) = \sum_j{}' e_j \int d\vec{r}^3\, \delta(\vec{r}-\vec{r}_j) = \sum_j{}' e_j$

* Evaluate the integral $I = \int_{-\infty}^{\infty} dx\, f(x)\, \delta(x^2-2)$. The zeros of $g(x) = x^2-2$ are at $x = \pm\sqrt{2}$, so that the integral can be written as the sum of two contributions:

$$I = \int_{\sqrt{2}-\varepsilon}^{\sqrt{2}+\varepsilon} dx\, f(x)\, \frac{\delta(x-\sqrt{2})}{\left|\frac{d(x^2-2)}{dx}\right|_{x=\sqrt{2}}} + \int_{-\sqrt{2}-\varepsilon}^{-\sqrt{2}+\varepsilon} dx\, f(x)\, \frac{\delta(x+\sqrt{2})}{\left|\frac{d(x^2-2)}{dx}\right|_{x=-\sqrt{2}}} =$$

$$= \int_{\sqrt{2}-\varepsilon}^{\sqrt{2}+\varepsilon} dx\, f(x)\, \frac{\delta(x-\sqrt{2})}{2\sqrt{2}} + \int_{-\sqrt{2}-\varepsilon}^{-\sqrt{2}+\varepsilon} dx\, f(x)\, \frac{\delta(x+\sqrt{2})}{2\sqrt{2}} = \frac{f(\sqrt{2}) + f(-\sqrt{2})}{2\sqrt{2}}$$

* Scattering of relativistic particles yields Feynman diagram contribution of the form $(c=1)$

$$\int d^4p\, \delta(p^2-m^2)\, f(p) = \int d^3p \int dp_0\, \delta(p_0^2 - \vec{p}^2 - m^2)\, f(p) =$$

$$= \int_{E>0} d^3p\, f(E,\vec{p})\, \frac{1}{2\sqrt{m^2+\vec{p}^2}} + \int_{E<0} d^3p\, f(E,\vec{p})\, \frac{1}{2\sqrt{m^2+\vec{p}^2}}$$

The physical meaning of $\delta(p^2-m^2)$ is that the particle of mass $m$ and four-momentum $p^\mu = (p_0, \vec{p})$ is on its mass shell because $p^2 = m^2$ is equivalent to $E = \pm\sqrt{m^2+\vec{p}^{\,2}}$, corresponding to particles and antiparticles.

Derivative of the delta-function, $\delta'(x)$: We have

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta'(x-x_0) = -\int_{-\infty}^{\infty} dx\, f'(x)\, \delta(x-x_0) = -f'(x_0)$$

Here we used integration by parts. The "surface" term does not contribute, since $\delta(x-x_0) f(x) \Big|_{-\infty}^{\infty}$ is zero by the definition delta function. The derivative of the Dirac delta function yields then "minus the derivative of the function $f(x)$ at the singular point".

Dirac delta in three dimensions:

In cartesian coordinates we have

$$f(0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, f(x,y,z)\, \delta(x)\, \delta(y)\, \delta(z)$$

which corresponds to three one-dimensional $\delta$-functions.

In polar coordinates we have

$$f(0) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r^2 f(\vec{r})\, \delta(\vec{r}) =$$

$$= \int_0^{2\pi} d\varphi\, \delta(\varphi) \int_0^{\pi} d\theta \sin\theta\, \delta(\cos\theta) \int_0^{\infty} dr\, r^2 \frac{\delta(r)}{r^2} f(\vec{r})$$

$$= \int_0^{2\pi} d\varphi\, \delta(\varphi) \int_{-1}^{1} d(\cos\theta)\, \delta(\cos\theta) \int_0^{\infty} dr\, \delta(r)\, f(\vec{r})$$

A few more examples: 

* Verify that the sequence $\delta_n(x) = \begin{cases} 0 & x<0 \\ n e^{-nx} & x>0 \end{cases}$

is a delta kernel. The singularity is at $+0$, the positive side of the origin. First we show that for $x>0$

$$\lim_{n\to\infty} n e^{-nx} = 0$$ , which is obvious because the exponential

dominates of a power. Next we show that the function is normalized:

$$\int_0^\infty dx\, n e^{-nx} = \int_0^\infty dy\, e^{-y} = 1 \quad , \quad y = nx$$

* Show that $x\delta'(x) = -\delta(x)$:

$$\int_{-\infty}^{\infty} dx\, x\,\delta'(x) f(x) = -\int_{-\infty}^{\infty} dx\, (x f(x))' \delta(x) = -\int_{-\infty}^{\infty} dx\, f(x)\delta(x) -$$

$$- \underbrace{\int_{-\infty}^{\infty} dx\, x f'(x)\delta(x)}_{0} = -\int_{-\infty}^{\infty} dx\, f(x)\delta(x) = -f(0) \qquad \checkmark$$