Mathematical Physics — PHZ 3113

Classwork 8 (February 27, 2013)

Cylindrical Coordinates

1. Use cylindrical coordinates to calculate the area of a circle of radius R.

Solution:

$$A = \int_{S_{\text{circle}}} d^2x \;=\; \int_{x^2+y^2\leq R^2} dx\, dy$$

$$= \int_0^{2\pi} d\phi \int_0^R \rho\, d\rho \;=\; 2\pi\,\frac{1}{2}\,R^2 \;=\; \pi\, R^2\,.$$

2. Calculate

$$\nabla \times \hat{z}\, \ln(\rho)$$

in cylindrical coordinates.

Solution:

$$\nabla \times \hat{z}\, \ln(\rho) \;=\; \hat{\rho} \times \hat{z}\,\frac{1}{\rho} \;=\; -\frac{\hat{\phi}}{\rho}\,.$$

3. Show Oersted's law

$$\oint \vec{H} \cdot d\vec{r} \;=\; I$$

for the magnetic potential

$$\vec{A} = -\hat{z}\,\frac{\mu_0\,I}{2\pi}\,\ln(\rho)\,,\quad \vec{B} \;=\; \nabla\times\vec{A}\,,$$
$$\vec{H} \;=\; \mu_0^{-1}\,\vec{B}\,.$$

Solution:

$$\vec{H} \;=\; \frac{I\,\hat{\phi}}{2\pi\,\rho}\,,$$
$$\oint \vec{H}\cdot d\vec{r} \;=\; \int_0^{2\pi}\rho\,d\phi\,\frac{I}{2\pi\,\rho} \;=\; I\,.$$

4. Find the acceleration $\vec{a}$ in cylindrical coordinates.

Solution:

$$\vec{v} = \dot{\rho}\,\hat{\rho} + \rho\,\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$
$$\vec{a} = \dot{\vec{v}} \;=\; \ddot{\rho}\,\hat{\rho} + \dot{\rho}\,\dot{\hat{\rho}} + \dot{\rho}\,\dot{\phi}\,\hat{\phi} + \rho\,\ddot{\phi}\,\hat{\phi} + \rho\,\dot{\phi}\,\dot{\hat{\phi}} + \ddot{z}\,\hat{z}\,,$$
$$\dot{\hat{\rho}} = \dot{\phi}\,\hat{\phi}\,,\quad \dot{\hat{\phi}} \;=\; -\dot{\phi}\,\hat{\rho}\,,$$
$$\vec{a} = \left(\ddot{\rho} - \rho\,\dot{\phi}^2\right)\,\hat{\rho} + \left(\rho\,\ddot{\phi} + 2\,\dot{\rho}\,\dot{\phi}\right)\,\hat{\phi} + \dot{z}\,\hat{z}\,.$$

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Classwork 9 (March 1, 2013)

Cylindrical Coordinates

5. Completion of the square: We have

$$x^2 + b\,x + c$$

and want this in the form

$$x'^{\,2} + c' \,.$$

What are the values of x' and c'?

Solution:

$$x' \;=\; x + \frac{b}{2}\,, \quad c' \;=\; c - \frac{b^2}{4}\,.$$

6. In cylindrical coordinates the equation of an ellipse is given by

$$\frac{p}{\rho} = 1 + \epsilon\,\cos(\phi)\,, \quad p > 0$$

with Cartesian coordinates $x = \rho\,\cos(\phi)$ and $y = \rho\,\sin(\phi)$. Assume $0 < \epsilon < 1$ for the eccentricity and transform the solution into the form

$$\frac{x'^2}{a^2} + \frac{y'2}{b^2} = 1 \ .$$

This means, derive the definitions of x', y', major half-axis a and minor half-axis b in terms of x, y, p and ϵ.

Solution: Using $\cos\theta = x/\rho$, the initial equation becomes

$$\begin{aligned} p \ &= \ \rho\,(1+\epsilon\ \cos\theta) \ = \ \rho\,(1+\epsilon\, x/\rho) \\ &= \ \rho + \epsilon\, x \ \ \text{or} \ \ \rho = p - \epsilon\, x \ . \end{aligned}$$

Squaring both sides of the last equation:

$$x^2 + y^2 = p^2 - 2\,p\,\epsilon\,x + \epsilon^2\,x^2 \ .$$

Bringing all terms with x or y to one side,

$$x^2\,(1-\epsilon^2) + 2\,p\,\epsilon\,x + y^2 = p^2 \ ,$$

$$x^2 + \frac{2\,p\,\epsilon}{1-\epsilon^2}\,x + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{1-\epsilon^2} \ ,$$

According to the recipe for completion of the square we substitute

$$x' = x + \frac{p\,\epsilon}{1-\epsilon^2}$$

and obtain

$$x'^{\,2} + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{1-\epsilon^2} + \left(\frac{p\,\epsilon}{1-\epsilon^2}\right)^2$$

$$= \frac{p^2\,(1-\epsilon^2)+p^2\,\epsilon^2}{(1-\epsilon^2)^2} = \frac{p^2}{(1-\epsilon^2)^2}\,.$$

Multiplying with $(1-\epsilon^2)^2/p^2$, this becomes

$$x'^{\,2}\,\frac{(1-\epsilon^2)^2}{p^2} + y^2\frac{1-\epsilon^2}{p^2} = 1\,.$$

With the definitions

$$a = \frac{p}{1-\epsilon^2} \quad \text{and} \quad b = \frac{p}{\sqrt{1-\epsilon^2}}$$

and $y' = y$ this reads

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1\,.$$

7. Use the definition

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\,.$$

to calculate the area of an ellipse. Hint: Make a substitution, so that it becomes reduced to the area of a circle.

Solution: We want to calculate

$$A = \int_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\leq 1} dx\,dy\,.$$

With the substitution $y = b\,y'/a$ this becomes

$$A = \frac{b}{a}\int_{x^2+y'^2\leq a^2} dx\,dy' = \frac{b}{a}\pi\,a^2 = \pi\,a\,b\,.$$