Mathematical Physics — PHZ 3113

Solutions Final (May 2, 2013)

Einstein's convention is used in the following problems and ϵ_{ijk} is the Levi-Civita tensor.

1. Calculate

$$\epsilon_{12k}\,\epsilon_{21k} = \quad (1)$$

$$\epsilon_{ij3}\,\epsilon_{ij3} = \quad (2)$$

$$\epsilon_{ijk}\,\epsilon_{ijk} = \quad (3)$$

Solution:

$$\begin{aligned} \epsilon_{12k}\,\epsilon_{21k} &= \epsilon_{123}\,\epsilon_{213} = -1\,, \\ \epsilon_{ij3}\,\epsilon_{ij3} &= \epsilon_{123}\,\epsilon_{123} + \epsilon_{213}\,\epsilon_{213} = +2\,, \\ \epsilon_{ijk}\,\epsilon_{ijk} &= 3! = +6\,. \end{aligned}$$

2. Rewrite the expression

$$\epsilon_{ijk}\,\epsilon_{klm}\,\hat{x}_i\,\partial_j\,(A_l\,B_m)\,, \quad (4)$$

where the $\hat{x}_i$ are Cartesian unit vectors and $\partial_j = \frac{\partial}{\partial x_j}$, into

(A) A vector product.

(B) A sum of scalar products.

Solution:

(A) The expression (4) is the definition of

$$\nabla \times \left(\vec{A} \times \vec{B}\right)\,.$$

(B) It can be transformed into a sum of scalar products:

$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{klm}\,\hat{x}_i\,\partial_j\,(A_l\,B_m) &= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\;\hat{x}_i\,\partial_j\,(A_l\,B_m) \\ &= \hat{x}_i\,\partial_j\,(A_i\,B_j) - \hat{x}_i\,\partial_j\,(A_j\,B_l) \end{aligned}$$

$$= \left(\vec{B}\cdot\nabla\right)\vec{A} + \vec{A}\left(\nabla\cdot\vec{B}\right) - \left(\vec{A}\cdot\nabla\right)\vec{B} - \vec{B}\left(\nabla\cdot\vec{A}\right).$$

3. Consider spherical coordinates. Calculate the time derivative

$$\dot{\hat{\theta}} = \frac{d\,\hat{\theta}}{d\,t}$$

of the unit vector $\hat{\theta}$ and express the result in spherical coordinates.

Solution: From the definitions

$$\begin{aligned} \hat{r} &= \sin(\theta)\;\cos(\phi)\,\hat{x} + \sin(\theta)\;\sin(\phi)\,\hat{y} + \cos(\theta)\,\hat{z}\,, \\ \hat{\theta} &= \cos(\theta)\;\cos(\phi)\,\hat{x} + \cos(\theta)\;\sin(\phi)\,\hat{y} - \sin(\theta)\,\hat{z}\,, \\ \hat{\phi} &= -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y} \end{aligned}$$

we get

$$\begin{aligned} \frac{\partial\,\hat{\theta}}{\partial\,r} &= 0\,, \\ \frac{\partial\,\hat{\theta}}{\partial\,\theta} &= -\sin(\theta)\cos(\phi)\,\hat{x} - \sin(\theta)\sin(\phi)\,\hat{y} \\ &\quad - \cos(\theta)\,\hat{z} \;=\; -\hat{r}\,, \\ \frac{\partial\,\hat{\theta}}{\partial\,\phi} &= -\cos(\theta)\;\sin(\phi)\,\hat{x} + \cos(\theta)\;\cos(\phi)\,\hat{y} \\ &= \cos(\theta)\,\hat{\phi}\,. \end{aligned}$$

Therefore,

$$\dot{\hat{\theta}} \;=\; \frac{\partial\,\hat{\theta}}{\partial\,\theta}\,\dot{\theta} + \frac{\partial\,\hat{\theta}}{\partial\,\phi}\,\dot{\phi} \;=\; -\dot{\theta}\,\hat{r} + \cos(\theta)\,\dot{\phi}\,\hat{\phi}\,.$$

4. Find the eigenvalues and (normalized) eigenvectors of the matrix

$$\begin{pmatrix} +1 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +1 \end{pmatrix}\,.$$

Solution: The characteristic equation of the eigenvalue problem is

$$\det \begin{vmatrix} +1-\lambda & -1 & 0 \\ -1 & +2-\lambda & -1 \\ 0 & -1 & +1-\lambda \end{vmatrix} \;=\; -\lambda^3 + 4\,\lambda^2 - 3\,\lambda \;=\; 0$$

and the first eigenvalue is seen to be

$$\lambda_1 \;=\; 0\,. \tag{5}$$

Dividing λ out gives the quadratic equation

$$\lambda^2 - 4\,\lambda + 3 \;=\; 0$$

from which the eigenvalues

$$\lambda_{2,3} \;=\; 2 \pm \sqrt{4-3} \;=\; 2 \pm 1 \;=\; \begin{cases} 1\,, \\ 3\,. \end{cases} \tag{6}$$

follow.

The eigenvector for $\lambda_1 = 0$ follows from

$$\begin{pmatrix} +1 & -1 & 0 \\ -1 & +2 & -1 \\ 0 & -1 & +1 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix} = 0\,.$$

and

$$\begin{aligned} v_1^1 - v_1^2 &= 0 \;\Rightarrow\; v_1^2 = v_2^1\,, \\ -v_1^2 + 2v_1^2 - v_1^3 &= 0 \\ -v_1^2 + v_1^3 &= 0 \;\Rightarrow v_1^3 = v_1^1 \end{aligned}$$

and the middle equation gives then $v_1^2 = v_1^1$. Up to a $\pm$ convention the first eigenvector becomes

$$\hat{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\,.$$

The eigenvector for $\lambda_2 = 1$ follows from

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & +1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix} = 0\,.$$

and

$$\begin{aligned} -v_2^2 &= 0 \;\Rightarrow\; v_2^2 = 0\,, \\ -v_2^2 + 0 - v_2^3 &= 0 \;\Rightarrow\; v_2^3 = -v_2^1 \\ -v_2^2 &= 0\,. \end{aligned}$$

Up to a $\pm$ convention the second eigenvector becomes

$$\hat{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix}\,.$$

The eigenvector for $\lambda_3 = 3$ follows from

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} v_3^1 \\ v_3^2 \\ v_3^3 \end{pmatrix} = 0\,.$$

and

$$\begin{aligned} -2v_3^1 - v_3^2 &= 0 \Rightarrow v_3^2 = -2v_3^1\,, \\ -v_3^1 - v_3^2 - v_3^3 &= 0 \Rightarrow v_3^3 = -v_3^1 - v_3^2 = -v_3^1 + 2v_3^1 = v_3^1 \\ -v_3^2 - 2v_3^3 &= 0\,. \end{aligned}$$

Up to a $\pm$ convention the third eigenvector becomes

$$\hat{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} +1 \\ -2 \\ +1 \end{pmatrix}\,.$$

Consistency check: $\hat{v}_i \cdot \hat{v}_j = \delta_{ij}$.