Mathematical Physics — PHZ 3113

Curl; Vector Integration Homework

(February 7, 2013)

The dual of the Euclidean electromagnetic field tensor is defined by

$$^*F_{\alpha\beta} = \frac{1}{2}\,\epsilon_{\alpha\beta\mu\nu}F_{\mu\nu} \qquad (1)$$

where the indices run from 1 to 4, the Einstein convention is used and $F_{\mu\nu}$ is an anti-symmetric (i.e., $F_{\mu\nu} = -F_{\mu\nu}$) rank two tensor

$$\left(F_{\mu\nu}\right) \;=\; \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix}.$$

1. Calculate the elements of $^*F_{\alpha\beta}$ in terms of the independent elements of $F_{\mu\nu}$ and write $^*F_{\alpha\beta}$ as a matrix.

2. The electromagnetic field tensor can be written in form of derivatives of a 4-potential

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu\,. \qquad (2)$$

Use this expression to proof the homogeneous Maxwell equations in their form

$$\partial_\alpha \, {}^*F_{\alpha\beta} \;=\; 0\,. \tag{3}$$

Remark: Therefore (2) implies that no magnetic monopoles exist, because (3) includes (upon identification of the $\vec{E}$ and $\vec{B}$ fields) the relation $\nabla \cdot \vec{B} = 0$.