

Mathematical Physics — PHZ 3113

Einstein Convention Homework 1

1. Use the 3D identity

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \quad (1)$$

to calculate

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{b}) \quad (2)$$

where $\hat{a}$ and $\hat{b}$ are unit vectors. Eliminate all $\hat{a} \cdot \hat{b}$ in favor of $\cos \theta$.

Solution: Let

$$\hat{a} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} \quad \text{and} \quad \hat{b} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{pmatrix}. \quad (3)$$

Then

$$\begin{aligned} (\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{b}) &= \epsilon_{ijk}\epsilon_{ilm}\tilde{a}_j\tilde{b}_k\tilde{a}_l\tilde{b}_m \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \tilde{a}_j\tilde{b}_k\tilde{a}_l\tilde{b}_m \\ &= (\tilde{a}_l\tilde{b}_m\tilde{a}_l\tilde{b}_m - \tilde{a}_m\tilde{b}_l\tilde{a}_l\tilde{b}_m) \end{aligned}$$

$$\begin{aligned}
&= (\hat{a} \cdot \hat{a}) (\hat{b} \cdot \hat{b}) - (\hat{a} \cdot \hat{b}) (\hat{b} \cdot \hat{a}) \\
&= 1 - \cos^2(\theta) = \sin^2(\theta). \tag{4}
\end{aligned}$$

Einstein Convention Homework 2

1. Use the identity (1) to eliminate the vector products from the expression

$$\vec{a} \times (\vec{b} \times \vec{c}) \tag{5}$$

Solution (compare book p.33):

$$\begin{aligned}
&\vec{a} \times (\vec{b} \times \vec{c}) \\
&= \epsilon_{ijk} \hat{x}_i a_j \epsilon_{klm} b_l c_m \\
&= \epsilon_{kij} \epsilon_{klm} \hat{x}_i a_j b_l c_m \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \hat{x}_i a_j b_l c_m \\
&= \hat{x}_i a_j b_i c_j - \hat{x}_i a_j b_j c_i \\
&= \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) .
\end{aligned}$$

2. Use the definition

$$(\vec{b} \times \vec{c})_i = \epsilon_{ijk} b_j c_k \tag{6}$$

of the i^{th} component of the vector product $\vec{b} \times \vec{c}$ to prove

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) . \quad (7)$$

Solution (compare book p.30):

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$= \epsilon_{jki} b_j c_k a_i = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= \epsilon_{kij} c_k a_i b_j = \vec{c} \cdot (\vec{a} \times \vec{b}) .$$