Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 12

33. Cylindrical waveguide, TE_{11} mode power transmission.

The cutoff frequency follows as the $d \to \infty$ limit from our discussion of the cylindrical cavity,

$$\omega_{11} = \frac{1.841\, c}{\sqrt{\mu\epsilon}\, R} = \frac{\gamma_{11}^{'}}{R}\,.$$

Using the time-averaged Poynting vector

$$\vec{S} \;=\; \frac{c}{8\pi} \mathrm{Re}\left(\vec{E} \times \overline{\vec{H}}\right)$$

and the TE mode

$$H^z = \psi\, e^{ikz}\,, \quad \left.\frac{\partial \psi}{\partial n}\right|_S = 0\,, \quad \vec{H}_t = \frac{ik}{\gamma^2}\, \nabla_t \psi\,, \quad \vec{E}_t = -Z\, \hat{z} \times \vec{H}_t\,,$$

where $Z = \mu\omega/(ck)$ is the wwave impedance for the TE mode, one finds

$$\vec{S} \;=\; \frac{\omega\, k\, \mu}{8\pi\, \gamma^4} \left[\hat{z}\, |\nabla_t \psi|^2 - i\, \frac{\gamma^2}{k}\, \overline{\psi}\, \nabla_t \psi\right]\,.$$

We are only interested in the longitudinal energy $\sim \hat{z}$ and the power flow is obtained by integrating over the cross-sectional area:

$$P \;=\; \int_A \hat{z} \cdot \vec{S} \;=\; \frac{\omega\, k\, \mu}{8\pi\, \gamma^4} \int_A (\overline{\nabla_t \psi}) \cdot (\nabla_t \psi)\, da\,.$$

By means of Green's first identity applied to two dimensions, P becomes

$$P \;=\; \frac{\omega\, k\, \mu}{8\pi\, \gamma^4} \left[\oint_C \overline{\psi}\, \frac{\partial \psi}{\partial n}\, dl - \int_A \overline{\psi}\, \nabla_t^2 \psi\, da\right]\,,$$

where the first integral is zero due to the BC. Using the wave equation for the second integral, the transmitted power for mode γ_λ^2 is

$$P \;=\; \frac{\omega\, k_\lambda\, \mu}{8\pi\, \gamma_\lambda^4} \int_A \overline{\psi}\, \gamma_\lambda^2\, \psi\, da \;=\; \frac{c\, \mu}{8\pi\, \sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda}\right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \int_A \overline{\psi}\, \psi\, da\,,$$

where in the last step γ_λ^2 has been eliminated in favor of the cutoff frequency, $\gamma_\lambda = \sqrt{\mu\epsilon}\, \omega_\lambda / c$ and $k_\lambda = \sqrt{\mu\epsilon} \sqrt{\omega^2 - \omega_\lambda^2}/c$. For the TE_{11} mode

$$P = \frac{c\, \mu}{8\pi\, \sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_{11}}\right)^2 \left(1 - \frac{\omega_{11}^2}{\omega^2}\right)^{1/2} \int_A da\, \overline{\psi}\, \psi$$

and

$$\psi = H_0\, J_1(\gamma'_{11}\,\rho)\, e^{i\,\phi}$$

for the mode in question. Thus

$$P = \frac{c\,\mu}{8\pi\,\sqrt{\mu\epsilon}}\,\left(\frac{\omega}{\omega_{11}}\right)^2\,\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{1/2}\,2\pi\,|H_0|^2\int_0^R \rho\,d\rho\,\left[J_1(\gamma'_{11}\,\rho)\right]^2\ .$$

Using the integral as given in the problem the final result is

$$P = \frac{c\,\mu}{8\,\sqrt{\mu\epsilon}}\,\left(\frac{\omega}{\omega_{11}}\right)^2\,\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{1/2}\,|H_0|^2\,R^2\,\left(1-\frac{1}{x'^2_{11}}\right)\,\left[J_1(\gamma'_{11}\,R)\right]^2\ .$$

34. Hertz vector.

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha\,, \qquad (A^\alpha) = \begin{pmatrix}\Phi\\ \vec{A}\end{pmatrix}\ .$$

In the Lorentz gauge $\partial_\alpha A^\alpha = 0$ holds and, therefore,

$$\partial_\alpha F^{\alpha\beta} = \Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = \Box A^\beta = \frac{4\pi}{c}\,J^\alpha\ .$$

In components this reads

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\,\rho\,, \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\,\vec{J}\,.$$

(1) Continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\,(\nabla\cdot\vec{p}) + \nabla\cdot\left(\frac{\partial\vec{p}}{\partial t}\right) = 0\ .$$

(2) Let $\Phi = a\,\left(\nabla\cdot\vec{\Pi}\right)$ and $\vec{A} = b\,\partial\vec{\Pi}/\partial t$. Then,

$$a\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left(\nabla\cdot\vec{\Pi}\right) \;=\; -a\,4\pi\,\nabla\cdot\vec{p} \;=\; a\,4\pi\,\rho \;=\; -\,4\pi\,\rho \;\Rightarrow\; a \;=\; -1\,,$$

$$b\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial\vec{\Pi}}{\partial t}\right) \;=\; -b\,4\pi\,\frac{\partial\vec{p}}{\partial t} \;=\; -b\,4\pi\,\vec{J} \;=\; -\frac{4\pi}{c}\,\vec{J} \;\Rightarrow\; b \;=\; \frac{1}{c}\,,$$

$$\Phi = -\left(\nabla\cdot\vec{\Pi}\right)\,, \qquad \vec{A} = \frac{1}{c}\,\frac{\partial\vec{\Pi}}{\partial t}\ .$$

The Lorentz condition holds:

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \nabla\cdot\vec{A} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\cdot\vec{\Pi}\right) + \frac{1}{c}\nabla\cdot\frac{\partial\vec{\Pi}}{\partial t} = 0\ .$$

(3) From the given Hertz vector we find

$$\Phi = -\nabla\cdot\vec{\Pi} = -p_0\,\cos(\theta)\,\frac{\partial}{\partial r}\left(r^{-1}\,e^{-i\,\omega\,t+i\,k\,r}\right) \approx -\frac{i\,k\,p_0}{r}\,\cos(\theta)\,e^{-i\,\omega\,t+i\,k\,r}\,,$$

$$\vec{A} = \frac{1}{c}\,\frac{\partial\vec{\Pi}}{\partial t} = -\frac{i\,\omega\,p_0}{c\,r}\,\hat{z}\,e^{-i\,\omega\,t+i\,k\,r}\,.$$

(5) The electric and magnetic fields are given by

$$\vec{E} = -\frac{1}{c}\,\frac{\partial\vec{A}}{\partial t} - \nabla\Phi\,, \qquad \vec{B} = \nabla\times\vec{A}\ .$$

Therefore, we have in the far field approximation (use ∇ in spherical coordinates)

$$\vec{E} \approx \frac{\omega^2\,p_0}{c^2\,r}\,\left(-\hat{z} + \cos(\theta)\,\hat{r}\right)\,e^{-i\,\omega\,t+i\,k\,r}\ .$$

Using $\hat{z} = \cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\theta}$ this is

$$\vec{E} \approx \frac{\omega^2\,p_0}{c^2\,r}\,\sin(\theta)\,\hat{\theta}\,e^{-i\,\omega\,t+i\,k\,r}\ .$$

For the magnetic field we have

$$\vec{B} = -\frac{i\,\omega\,p_0}{c}\,\hat{r}\times\hat{z}\,\frac{\partial}{\partial r}\,\frac{e^{-i\,\omega\,t+i\,k\,r}}{r} \approx -\frac{\omega\,k\,p_0}{c\,r}\,\sin(\theta)\,\hat{\phi}\,e^{-i\,\omega\,t+i\,k\,r}\ .$$

The $\vec{E}$ and $\vec{B}$ fields describe electric dipole radiation.

35. Vector potential of two current loops.

We define

$$x_0 = \cos(\theta_0) = \frac{d}{r_0} \quad \text{with} \quad r_0 = \sqrt{R^2 + d^2} \quad \text{and} \quad x' = \cos(\theta') \ .$$

The currents flow in $\hat{\phi}$ direction and their densities are given by

$$J_\phi^{(1,2)} = \pm I \, \sin(\theta') \, \delta(\cos\theta_0 \mp \cos\theta') \, r_0^{-1} \, \delta(r' - r_0) \ .$$

Note the (correct) normalization

$$\int_0^\infty r' dr' \int_0^\pi d\theta' \, J_\phi^{(1,2)} = \pm I$$

for the currents. Using Cartesian coordinates we have

$$\vec{J}^{(1,2)} = -J_\phi^{1,2} \, \sin(\phi') \, \hat{x} + J_\phi^{1,2} \, \cos(\phi') \, \hat{y} \ .$$

(a) For the calculation of the vector potential $\vec{A}$ we exploit cylindrical symmetry and choose the observation point at $\phi = 0$, so that the $\sin(\phi') \, \hat{x}$ component does not contribute. It remains to calculate the $\hat{y}$ component, $\hat{\phi} = \hat{y}$, $r \gg r'$ now:

$$A_\phi = \frac{1}{c} \int_0^{2\pi} d\phi' \, \cos(\phi') \int_{-1}^{+1} d\cos\theta' \int_0^\infty r'^2 dr' \, J_\phi \, \frac{e^{i\,k\,|\vec{x} - \vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|} \, ,$$

$$\frac{e^{i\,k\,|\vec{x} - \vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|} = 4\pi \, i \, k \sum_{l=0}^{\infty} j_l(kr') \, h_l^+(kr) \sum_{m=-l}^{+l} Y_l^{\ m}(\theta, \phi) \, \overline{Y}_l^{\ m}(\theta', \phi') \, .$$

After the r'-integration we have

$$A_\phi = 4\pi \, i \, k \frac{I \, r_0}{c} \sum_{l=0}^{\infty} j_l(kr_0) \, h_l^+(kr) \sum_{m=-l}^{+l} Y_l^{\ m}(\theta, \phi)$$
$$\int_{-1}^{+1} d\cos\theta' \int_0^{2\pi} d\phi' \, \cos(\phi') \, \overline{Y}_l^{\ m}(\theta', \phi') \, .$$

As

$$\int_0^{2\pi} d\phi' \, \cos(\phi') = 0$$

holds, it follows that ${Y_0}^0$, ${Y_1}^0$ and ${Y_2}^0$ do not contribute. For ${Y_1}^1 = -\sqrt{3}\,\sin\theta\, e^{i\phi}/\sqrt{8\pi}$, and similarly Y_1^{-1}, we have with $x' = \cos\theta'$

$$\int_{-1}^{+1} dx'\,\sqrt{1-x'^2}\;[\delta(x_0-x')-\delta(x_0+x')] \;=\; 0\,,$$

and for ${Y_2}^2$

$$\int_0^{2\pi} d\phi'\,\cos(\phi')\,e^{i\,2,\phi'} = 0\,.$$

So,

$${Y_2}^1 = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\,e^{i\,\phi},$$

and similarly Y_2^{-1}, give identical leading contributions, which are electric quadrupoles. We have:

$$\int_0^{2\pi} d\phi'\,\cos(\phi')\,e^{-i\,\phi'} = \pi\,,$$

$$\int_{-1}^{+1} dx'\,\sqrt{1-x'^2}\,x\;[\delta(x_0-x')-\delta(x_0+x')]$$
$$= 2\,\sqrt{1-x_0^2}\,x_0 \;=\; 2\,\sin(\theta_0)\,\cos(\theta_0)\,,$$

and, adding ${Y_2}^1$ and Y_2^{-1} contributions, in this approximation the potential is

$$A_\phi = -\frac{I\,r_0}{c}\,\sqrt{30\pi}\,\sin(\theta_0)\,\cos(\theta_0)\,i\,k\,4\pi\,j_2(kr_0)\,h_2^+(kr)\,{Y_2}^1(\theta,\phi=0)\,.$$

In the $r\to\infty$ far field approximation we have

$$h_2^+(kr) = i\,\frac{e^{i\,k\,r}}{r}$$

and, consequently,

$$A_\phi(r,\theta) = A_\phi^0\,\sin(\theta)\,\cos(\theta)\,\frac{e^{i\,k\,r}}{r}$$

with

$$A_\phi^0 \;=\; -30\pi\,k\,\frac{I\,r_0}{c}\,j_2(kr_0)\,\sin(\theta_0)\,\cos(\theta_0)\,.$$