

Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 2:

2. Magnetic dipoles.

The magnetic dipole approximation holds

$$\vec{B}(\vec{r}) = \frac{3\hat{r}(\hat{r} \cdot \vec{m}) - \vec{m}}{|\vec{r}|^3}, \quad \hat{r} = \frac{\vec{r}}{r}, \quad r = |\vec{r}|.$$

- (1) Now we consider \vec{m}_1 in the field of \vec{m}_2 , $\vec{r} = \vec{r}_1 - \vec{r}_2$:

$$U = -\vec{m}_1 \cdot \vec{B}_2 = - \left(\frac{3(\vec{m}_1 \cdot \hat{r})(\hat{r} \cdot \vec{m}_2) - \vec{m}_1 \cdot \vec{m}_2}{|\vec{r}|^3} \right),$$

- (2) The force acting on either of the magnets follows from

$$\vec{F} = -\nabla U, \quad \nabla|\vec{r}| = \hat{r} = \frac{\vec{r}}{r}.$$

We keep the position of \vec{m}_2 fixed and calculate the force acting on \vec{m}_1 at the position \vec{r} :

$$\begin{aligned} \vec{F} &= +\nabla \frac{3(\vec{m}_1 \cdot \vec{r})(\vec{r} \cdot \vec{m}_2)}{|\vec{r}|^5} - \nabla \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{r}|^3} \\ &= \frac{3\vec{m}_1(\vec{r} \cdot \vec{m}_2)}{|\vec{r}|^5} + \frac{3\vec{m}_2(\vec{r} \cdot \vec{m}_1)}{|\vec{r}|^5} - 15\vec{r} \frac{(\vec{m}_1 \cdot \vec{r})(\vec{r} \cdot \vec{m}_2)}{|\vec{r}|^7} + 3\vec{r} \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{r}|^5}. \end{aligned}$$

- (3) The rest position is at $U = U_{\min}$. Let us choose coordinates, so that $\hat{r} = \hat{z}$. Then

$$U = \frac{-2m_1^z m_2^z + m_1^x m_2^x + m_1^y m_2^y}{|\vec{r}|^3}.$$

We find negative contributions for $\text{sign}(m_1^z) = \text{sign}(m_2^z)$, $\text{sign}(m_1^x) = -\text{sign}(m_2^x)$ and $\text{sign}(m_1^y) = -\text{sign}(m_2^y)$, which are therefore assumed to be chosen by the system. In particular, this leads to

$$\vec{m}_2^\perp = -\alpha \vec{m}_1^\perp, \quad \alpha > 0$$

for the components perpendicular to \hat{z} (i.e. they are vectors in the x - y plane). For $i = 1, 2$ let $m_i^\perp = |\vec{m}_i^\perp|$. Then (all terms give negative contributions)

$$U = \frac{-2m_1^z m_2^z - m_1^\perp m_2^\perp}{|\vec{r}|^3}.$$

With $m_i = |\vec{m}_i|$ for $i = 1, 2$ we can write

$$m_1^z = m_1 \cos(\alpha), \quad m_1^\perp = m_1 \sin(\alpha) \quad \text{and} \quad m_2^z = m_2 \cos(\beta), \quad m_2^\perp = m_2 \sin(\beta)$$

and obtain

$$U = \frac{-2 m_1 m_2 \cos(\alpha) \cos(\beta) - m_1 m_2 \sin(\alpha) \sin(\beta)}{|\vec{r}|^3}.$$

So, we have to minimize $-2 \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$. Using well-known trigonometric equations, this becomes

$$-\frac{3}{2} \cos(\alpha - \beta) - \frac{1}{2} \cos(\alpha + \beta).$$

As the maximum of $\cos(\phi)$ is at $\phi = 0$, the only solution for the minimum of U is $\alpha = \beta = 0$. Hence, \vec{m}_1 is parallel to \vec{m}_2 along the \hat{r} axis.

In this configuration there is an attractive force of magnitude

$$F_{\min} = |3 + 3 - 15 + 3| \frac{m_1 m_2}{r^4} = 6 \frac{m_1 m_2}{r^4}$$

between the dipoles. The magnetic field of this configuration is

$$\begin{aligned} \vec{B}(\vec{r}) &= \sum_{i=1}^2 \frac{3 (\vec{r} - z_i \hat{z}) m_i \hat{z} \cdot (\vec{r} - z_i \hat{z}) - m_i \hat{z} (\vec{r} - z_i \hat{z})^2}{|\vec{r} - z_i \hat{z}|^5} \\ &= \sum_{i=1}^2 \frac{3 (\vec{r} - z_i \hat{z}) m_i (r \cos \theta - z_i) - m_i \hat{z} [r^2 - 2 r z_i \cos \theta + (z_i)^2]}{[r^2 - 2 r z_i \cos \theta + (z_i)^2]^{5/2}}. \end{aligned}$$

3. Magnetic moment of a disk.

- (1) The charge density on the perimeter is $\lambda = q/(2\pi R)$, so the current density is (with $r = |\vec{r}|$)

$$\vec{J}(\vec{r}) = \omega R \lambda \delta(\cos \theta) R^{-1} \delta(r - R) \hat{\phi},$$

and the current becomes

$$\vec{I} = \int_0^\pi d\theta \int_0^\infty r dr \vec{J}(\vec{r}) = \omega R \lambda \hat{\phi}.$$

The magnetic moment is now

$$\begin{aligned}\vec{\mu} &= \frac{1}{2c} \int \vec{r} \times \vec{J} d^3x \\ &= \frac{2\pi}{2c} \omega \lambda (-\hat{\theta}) \int_{-1}^{+1} d\cos\theta \delta(\cos\theta) \int_0^\infty r^3 dr \delta(r-R) \\ &= \frac{\pi}{c} \omega R^3 \lambda \hat{z} = \frac{1}{2c} \omega R^2 q \hat{z}.\end{aligned}$$

(2) Let σ_m be the mass density of the disk. The angular momentum is

$$\vec{L} = L_z, \quad L_z = 2\pi \sigma_m \omega \int_0^R r^3 dr = \frac{\pi}{2} \sigma_m \omega R^4,$$

and the total mass is given by

$$M = 2\pi \int_0^R r dr \sigma_m = \pi R^2 \sigma_m \Rightarrow L_z = \frac{M}{2} \omega R^2.$$

Therefore, the magnetic moment can be written

$$\mu_z = g \frac{q}{2Mc} L_z \quad \text{with } g = 2$$

which is the g factor of the electron.