

The solution by taking derivative of work with respect to angle  $\beta$  is correct, but the students' solution has a mistake. They define

$$d\vec{\tau} = \vec{r} \times d\vec{F} \quad (1)$$

as the differential torque on each differential of area on one of the plates and also use  $d\vec{F} = \sigma \vec{E}$  and they use the same electric field that they have found in the previous part. Using this electric field is causing the problem.

The problem can be easily seen in a parallel plate capacitor when we calculate the force. So let's see the result of differentiating the work

$$\begin{aligned} U &= \frac{1}{2}QV \\ V &= ED \\ E &= \frac{Q}{\epsilon_0 A} \\ F &= -\frac{dU}{dD} = -\frac{Q^2}{2\epsilon_0 A}. \end{aligned}$$

On the other hand if we try to use integrating force on each differential of charge on the surface

$$F = \int dq \vec{E} = -Q \frac{Q}{\epsilon_0 A} \quad (2)$$

This result is larger by a factor of 2 and that is because the electric field in the integration is the total electric field which is the sum of the electric field of the two plates,  $\vec{E} = \vec{E}_1 + \vec{E}_2$ . The correct way is to calculate the force on each charge element due to the electric field of the other plate only and that gives the correct result.

$$F = \int dq \vec{E}_1 = -Q \frac{Q}{2\epsilon_0 A} \quad (3)$$

It is the same thing that causes the discrepancy in the homework. The correct and easy way is to go with your solution.