Electrodynamics B (PHY 5347): Spring 2017 Final, May 2.

1. **Magnetic moment of a sphere (33%).**

   A sphere with radius R rotates with constant angular velocity ω about the z axis. A uniform charge distribution is fixed on the surface. The total charge is q. Calculate the magnetic moment $\vec{\mu}$.

2. **Principal value integrals and Green functions of the wave equation (33%).**

   Green functions of the Klein-Gordon wave equation are defined by the equation

$$\left(\Box + m^2\right) G(x,y) \;=\; \delta^{(4)}(x-y)\,.$$

   After Fourier transformation solutions are found in the form

$$G(x-y) \;=\; \frac{1}{(2\pi)^3}\int d^3p\, I_p\,, \quad I_p \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{du\, e^{-i\,u\,\tau}}{\omega_p^2-u^2}$$

   with

$$\omega_p = +\sqrt{\vec{p}^{\;2}+m^2}\,, \quad p^0 = u \;\text{ and }\; \tau = x^0 - y^0\,.$$

   Closing in the complex $w = u + iv$ plane, one obtains Feynman (F), Dyson (D), advanced (a) and retarded (r) Green functions, depending on the convention for the poles in the integrand of I_p. For instance,

$$G_a(x-y) \;=\; \frac{1}{(2\pi)^3}\int d^3p\, I_p^a \quad\text{and}\quad G_r(x-y) \;=\; \frac{1}{(2\pi)^3}\int d^3p\, I_p^r\,,$$

$$I_p^r \;=\; \begin{cases} -\frac{i}{2\omega_p}\left(e^{+i\omega_p\tau}-e^{-i\omega_p\tau}\right) \text{ for } \tau>0, \\ 0 \text{ for } \tau<0; \end{cases} \qquad I_p^a \;=\; \begin{cases} 0 \text{ for } \tau>0; \\ \frac{i}{2\omega_p}\left(e^{+i\omega_p\tau}-e^{-i\omega_p\tau}\right) \text{ for } \tau<0, \end{cases}$$

   for the retarded and advanced choices.

   Calculate the principal value integral

$$I_p^P \;=\; \frac{1}{2\pi}\,P\int_{-\infty}^{+\infty}\frac{du\, e^{-i\,u\,\tau}}{\omega_p^2-u^2}$$

   for $\tau>0$ and $\tau<0$. Relate the $\tau>0$ result to I_p^r and the $\tau<0$ result to I_p^a.

3. **Far field approximation (34%).**

Use the vector potential

$$\vec{A}_\phi(r,\theta) = A^0_\phi \, \sin(\theta)\, \cos(\theta)\, \frac{e^{i\,k\,r - i\,\omega\,t}}{r}\, \hat{\phi}\ .$$

to calculate the following quantities in the far field approximation:

(a) The magnetic field.

(b) The electric field.

(c) The time-averaged Poynting vector.

(d) The angular intensity distribution.

(e) The maxima of the angular intensity distribution.

(f) The maxima of the angular intensity distributions divided by the average of the angular intensity distribution over the surface of the sphere that contains the maxima.