Electrodynamics B (PHY 5347): Spring 2017 Solutions for the Test on Homework.

1. **Magnetism in matter.**

   (a) For this magnetostatics problem with no free currents the relevant Maxwell equations are

   $$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{H} = 0$$

   with the associated boundary conditions (BCs)

   $$\hat{r} \cdot \left(\vec{B}_2 - \vec{B}_1\right)\Big|_{r=a,b} = 0\,, \qquad \hat{r} \times \left(\vec{H}_2 - \vec{H}_1\right)\Big|_{r=a,b} = 0\,.$$

   (b) Since $\nabla \times \vec{H} = 0$ we have $\vec{H} = -\nabla\Phi$ and thus $0 = \nabla \cdot \vec{B} = \nabla \cdot \left(\mu \vec{H}\right)$.

   From the BCs we get

   $$0 = \hat{r} \cdot \left(\vec{B}_2 - \vec{B}_1\right) = \hat{r} \cdot \left(\mu_2 \vec{H}_2 - \mu_1 \vec{H}_1\right) = -\mu_2 \frac{\partial \Phi_2}{\partial r} + \mu_1 \frac{\partial \Phi_1}{\partial r}$$

   and (because of azimuthal symmetry $\Phi(\vec{x}) = \Phi(r,\theta)$ holds)

   $$0 = \hat{r} \times \left(\vec{H}_2 - \vec{H}_1\right) = \hat{r} \times \left(-\nabla\Phi_2 + \nabla\Phi_1\right) = \frac{\hat{r} \times \hat{\theta}}{r} \left(-\frac{\partial \Phi_2}{\partial \theta} + \frac{\partial \Phi_1}{\partial \theta}\right)\,.$$

   Finally, the behavior of the field at infinity:

   $$\lim_{r\to\infty} \vec{B}(\vec{r}) = \vec{B}_0 = B_0\,\hat{z} \;\Rightarrow\; \lim_{r\to\infty} \Phi(\vec{r}) = -B_0\,z = -B_0\,r\,\cos\theta = -B_0\,r\,P_1(\cos\theta)\,.$$

2. **Covariant retarded Green function.**

   With $\tau = x^0 - y^0$ the zeros of $\delta[(x-y)^2]$ are at

   $$\tau^0 = \pm\sqrt{(\vec{x}-\vec{y}\,)^2} = \pm|\vec{x}-\vec{y}\,|\,.$$

   Let now $f(\tau) = \tau^2 - (\vec{x}-\vec{y}\,)^2$, so that

   $$\frac{df}{d\tau}\bigg|_{\tau=\tau^0} = f'(\tau^0) = 2\,\tau^0$$

   holds. Therefore,

   $$\begin{aligned}
   \delta[(x-y)^2] &= \frac{1}{2\,|f'(\tau^0)|}\left[\delta(\tau - |\vec{x}-\vec{y}\,|) + \delta(\tau + |\vec{x}-\vec{y}\,|)\right] \\
   &= \frac{1}{2\,|\vec{x}-\vec{y}\,|}\left[\delta(x^0 - y^0 - |\vec{x}-\vec{y}\,|) + \delta(x^0 - y^0 + |\vec{x}-\vec{y}\,|)\right]
   \end{aligned}$$

and the identity of the two expressions for the Green function follows:

$$\frac{1}{2\pi}\,\theta(x^0-y^0)\,\delta((x-y)^2)=\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}\,|)}{4\pi\,|\vec{x}-\vec{y}\,|}\,.$$

Charge conjugation, parity and time reversal in electrodynamics.

(1) Using the given values $(-,+,+)$ for ρ as input, our (C,P,T) results are obtained as follows:

$$\begin{aligned}
1.\ A^0\sim\rho &\ \Rightarrow\ (-,+,+)\,,\\
2.\ \vec{J}\sim\vec{v}\,\rho &\ \Rightarrow\ (-,-,-)\,,\\
3.\ \vec{A}\sim\vec{J} &\ \Rightarrow\ (-,-,-)\,,\\
4.\ E^i=F^{i0}=\partial^i A^0-\partial^0 A^i &\ \Rightarrow\ (-,-,+)\,,\\
5.\ B^k\sim F^{ij}=\partial^i A^j-\partial^j A^i &\ \Rightarrow\ (-,+,-)\,.
\end{aligned}$$

(2) From the explicit matrix forms we find

$$F^{\alpha\beta}\,F_{\alpha\beta}=-2\,\vec{E}^{\,2}+2\,\vec{B}^{\,2}\quad\text{and}\quad {}^*F^{\alpha\beta}\,F_{\alpha\beta}=-4\,\vec{E}\cdot\vec{B}\,.$$

Using the (C,P,T) results above, we find $(+,+,+)$ for $F^{\alpha\beta}\,F_{\alpha\beta}$ (proper scalar), and $(+,-,-)$ for ${}^*F^{\alpha\beta}\,F_{\alpha\beta}$ (pseudoscalar).

TM waves in a rectangular wave guide.

(a)

$$(\nabla_t^2+\gamma^2)\,E^z=0\,,\qquad E^z|_S=0\,,$$

$$E^z=E^0\,\sin\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{b}\right)\,e^{i\,k\,z}\,e^{-i\,\omega\,t}\,,$$

$$\gamma_{n\,m}^2=\pi^2\,\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]\,,\quad m=1,2,\dots\,,\quad n=1,2,\dots\,,$$

(b)

$$\gamma_{1\,1}^2=\pi^2\,\left(a^{-2}+b^{-2}\right)\,,\qquad \omega_{1\,1}=\frac{c\,\pi}{\sqrt{\mu\,\epsilon}}\,\sqrt{a^{-2}+b^{-2}}$$