Electrodynamics B (PHY 5347) Winter/Spring 2014 Solutions for the Midterm.

1. **Bar sliding in a magnetic field.**

   (a) Since the B field is constant the magnetic flux is

$$\Phi = B\,w\,(l - v\,t) \quad \text{and} \quad \frac{d\Phi}{dt} = -B\,w\,v\,.$$

   Therefore, the induced emf in the loop is

$$\varepsilon_{\rm emf} = -\frac{1}{c}\,\frac{d\phi}{dt} = c^{-1}\,B\,w\,v\,.$$

   The same result is obtained using the Lorentz force:

$$\varepsilon_{\rm emf} = -\frac{1}{c}\oint_C \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l} = -\frac{B\,v}{c}\int_0^w dl = c^{-1}\,B\,w\,v\,.$$

   (b) Since the resistance of the loop is $R(t) = [2w + 2(l - vt)]\,r$, we have the current

$$I(t) = \frac{\varepsilon_{\rm emf}}{R(t)} = \frac{B\,w\,v}{c\,r\,[2w + 2(l - vt)]} \quad \text{for} \quad vt \le l\,.$$

2. **Four-potential of a moving point particle.**

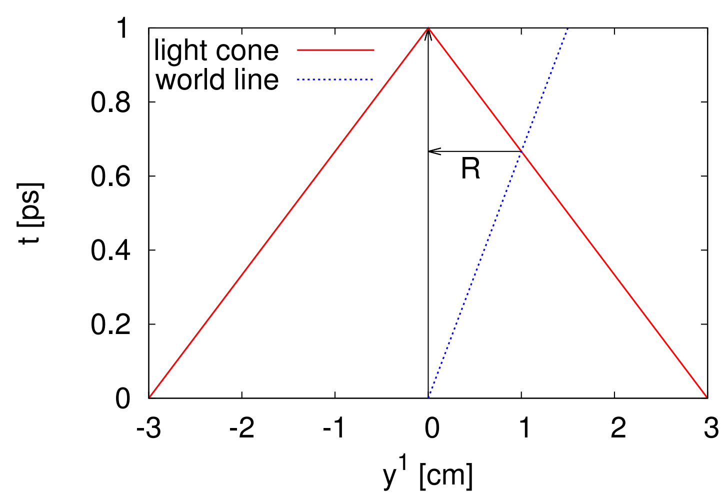

FIG. 1: Backward light cone and worldline in Minkowski space.

The worldline is given by $y^1 = 0.5\,y^0 = 0.5\,ct$. As light travels in one $[ps]$ by $3\,[dm]$, the equation for the relevant part of the backward light cone is $y^1 = 3\,[dm] - y^0$. Both meet when times and positions agree. Subtracting the first from the second equation gives

$$0 = 3\,[dm] - 1.5\,y^0 = 3\,[dm] - 1.5 \times 3 \times 10^9\,[dm/s]\,t\,,$$

which solves for

$$t \;=\; \frac{2}{3} \times 10^{-9}\,[s] \;=\; \frac{2}{3}\,[ps] \;\;\text{and}\;\; R = \beta\,c\,t = 1\;[dm]\,.$$

We find the potentials now from the equations

$$\Phi\,(t,\vec{x}) = \left[\frac{q}{\left(1-\vec{\beta}\cdot\hat{n}\right)\,R}\right]_{\rm ret}, \quad \vec{A}\,(t,\vec{x}) = \left[\frac{q\,\vec{\beta}}{\left(1-\vec{\beta}\cdot\hat{n}\right)\,R}\right]_{\rm ret}.$$

We have $\vec{\beta} = \beta\,\hat{y}^1 = (1/2)\,\hat{y}_1$ and, as the point charge moves away from the observer, $\vec{R} = R\,\hat{n} = -R\,\hat{y}^1$, i.e., $\hat{n} = -\hat{y}_1$. Therefore,

$$\begin{aligned}
\Phi\left(1\,[ps],\vec{0}\right) &= \frac{q_e}{(1+1/2)\,R} = \frac{2}{3}\,[q_e/dm]\,,\\
\vec{A}\left(1\,[ps],\vec{0}\right) &= \frac{q_e\,(1/2)\,\hat{y}^1}{(1+1/2)\,R} = \frac{1}{3}\,\hat{y}^1[q_e/dm]\,.
\end{aligned}$$

3. **Euler-Lagrange equation.**

First,

$$\frac{\partial\mathcal{L}}{\partial h_{\alpha\beta}} = \partial_\gamma\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\gamma h_{\alpha\beta}\right)} = 0 \;\Rightarrow\; h^{\alpha\beta} + \partial^\beta\,A^\alpha - \partial^\alpha\,A^\beta = 0\;.$$

So, $h^{\alpha\beta}$ is antisymmetric and agrees with the electromagnetic field tensor $h^{\alpha\beta} = F^{\alpha\beta}$. Second,

$$\partial_\alpha\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\alpha A_\beta\right)} = \frac{\partial\mathcal{L}}{\partial A_\beta} \;\Rightarrow\; \partial_\alpha\,\left(h^{\beta\alpha} - h^{\alpha\beta}\right) = -\frac{8\pi}{c}\,J^\beta\;.$$

Using the antisymmetry of $h^{\alpha\beta}$, this reads

$$\partial_\alpha\,h^{\alpha\beta} = \frac{4\pi}{c}\,J^\beta\;.$$

The Lagrangian has led to the inhomogenous Maxwell equations.