Electrodynamics B (PHY 5347): Spring 2017 Solutions for the Final.

1. **Magnetic moment of a sphere.**

The charge density on the surface is $\sigma = q/(4\pi\, R^2)$, so the current on the sphere becomes (with $r = |\vec{r}|$)

$$\vec{J}(\vec{r}) = \omega\, r\, \sin(\theta)\, \sigma\, \delta(r-R)\, \hat{\phi}\ .$$

The magnetic moment is then

$$\vec{\mu} = \frac{1}{2c}\int \vec{r}\times\vec{J}\, d^3x = \frac{1}{2c}\int \omega^2\, r^2\, \sin(\theta)\, \sigma\, \delta(r-R)\, (-\hat{\theta})\, d^3x\ .$$

By symmetry reasons only $\mu_z \neq 0$. The projection on the z-axis is found using $\hat{z}\cdot\hat{\theta} = -\sin(\theta)$. So,

$$\mu_z = \frac{2\pi}{2c}\,\omega\, R^4\, \sigma \int_0^{\pi} \sin^3(\theta)\, d\theta = \frac{\pi}{c}\,\omega\, R^4\, \sigma \int_{-1}^{+1} \left(1-x^2\right)\, dx\, .$$

The integral has the value 4/3 and our final result is

$$\mu_z = \frac{4\pi}{3c}\,\omega\, R^4\, \sigma = \frac{\omega\, R^2\, q}{3c}\ .$$

2. **Principal value integrals and Green functions for the wave equation.**

For $\tau > 0$ we have to close in the lower half-plane, where we get no contribution from closing ($e^{-i\,i\,v\,\tau} = e^{v\,\tau} \Rightarrow v < 0$ required). The integration path of Fig. 1

leads to the retarded Green function and we have

$$\lim_{\eta\to 0} I_p(\eta) \ = \ I_p^r \ = \ -\frac{i}{2\omega_p}\left(e^{+i\omega_p\tau} - e^{-i\omega\tau}\right) \ =$$

$$\frac{1}{2\pi}\lim_{\eta\to 0}\left[\int_{-\infty}^{-\omega_p-\eta} du\, \frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2} + \int_{-\omega_p+\eta}^{\omega_p-\eta} du\, \frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2} + \int_{\omega_p+\eta}^{\infty} du\, \frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2}\right]$$
$$+\frac{1}{2\pi}\lim_{\eta\to 0}\left[\int_{C_+(-\omega_p,\eta)} dw\, \frac{e^{-i\,w\,\tau}}{\omega_p^2-w^2} + \int_{C_+(+\omega_p,\eta)} dw\, \frac{e^{-i\,w\,\tau}}{\omega_p^2-w^2}\right]\ ,$$

where $C_+(-\omega_p,\eta)$ denotes the clockwise half-circle of radius η about $-\omega_p$ and $C_+(+\omega_p,\eta)$ the clockwise half-circle of radius η about $+\omega_p$. In the limit $\eta\to 0$ integration over the half-circles gives half the contribution from the residue theorem for

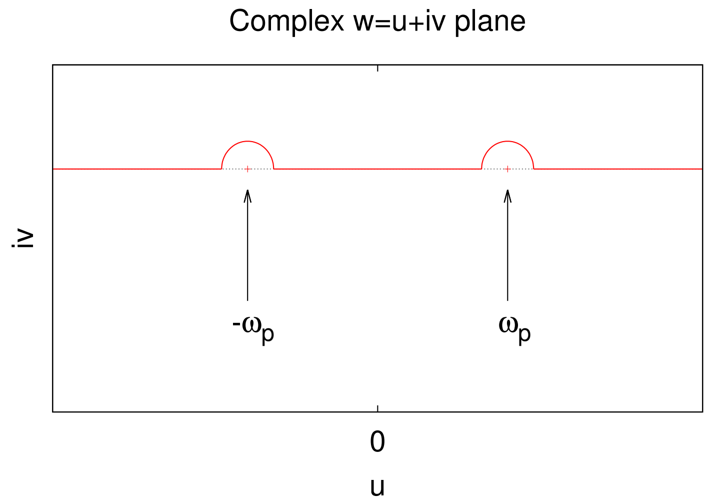

FIG. 1: A possible integration path $I_p(\eta)$, where $\eta > 0$ is the radius of the half-circles, and we close in the lower plane.

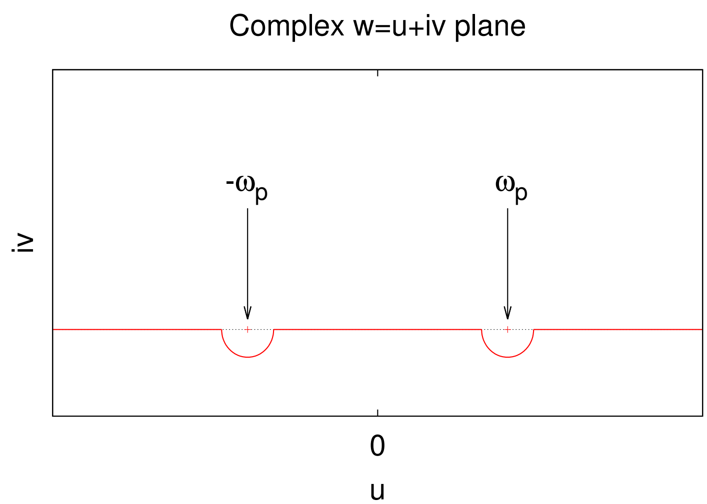

FIG. 2: Another integration path $I_p(\eta)$, where $\eta > 0$ is the radius of the half-circles.

the entire integration path, and the integrals on the real axis add up to the principal value definition. So we have

$$I_p^r \;=\; \frac{1}{2\pi}\,P\int_{-\infty}^{+\infty} du\,\frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2} + \frac{1}{2}\,I_p^r \;\Rightarrow\; \frac{1}{2\pi}\,P\int_{-\infty}^{+\infty} du\,\frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2} = \frac{1}{2}\,I_p^r\,.$$

For $\tau < 0$ we have to close in the upper half-plane and the integration path of Fig. 1 gives

$$0 \;=\; \frac{1}{2\pi}\,P\int_{-\infty}^{+\infty} du\,\frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2} - \frac{i}{4\omega_p}\left(e^{+i\omega_p\tau}-e^{-i\omega_p\tau}\right) \quad\Rightarrow$$

$$\frac{1}{2\pi}\,P\int_{-\infty}^{+\infty}du\,\frac{e^{-i\,u\,\tau}}{\omega_p^2-u^2}=\frac{i}{4\omega_p}\,\left(e^{+i\omega_p\tau}-e^{-i\omega_p\tau}\right)\;=\;\frac{1}{2}\,I_p^a\,.$$

Alternatively, one could have used the integration path of Fig. 2 and obtains the same results.

3. **Far field approximation of two current loops.**

Using the given potential $\vec{A}$, we have (a)

$$\vec{B}=i\,k\,\hat{r}\times\vec{A}=-i\,k\,\hat{\theta}\,A_\phi^0\,\sin(\theta)\,\cos(\theta)\,\frac{e^{i\,k\,r-i\,\omega\,t}}{r}$$

and (b)

$$\vec{E}=-\hat{r}\times\vec{B}=i\,k\,\hat{\phi}\,A_\phi^0\,\sin(\theta)\,\cos(\theta)\,\frac{e^{i\,k\,r-i\,\omega\,t}}{r}.$$

(c) The time-averaged Poynting vector becomes ($\hat{\phi}\times\hat{\theta}=-\hat{r}$)

$$\vec{S}_{\rm av}=\frac{c}{8\pi}\,\vec{E}\times\vec{B}^*=\frac{c}{8\pi}\,\hat{r}\,k^2\,\left|A_\phi^0\right|^2\,\frac{\sin^2(\theta)\,\cos^2(\theta)}{r^2}\,.$$

(d) Therefore, the angular intensity distribution is

$$\left(\frac{dP}{d\Omega}\right)=r^2\,\hat{r}\cdot\vec{S}_{\rm av}=\frac{c}{8\pi}\,k^2\,\left|A_\phi^0\right|\,\sin^2(\theta)\,cos^2(\theta)\,,$$

which may be plotted in a polar intensity diagram. This is the quadrupole case (see the figure of the lecture notes).

(e) The maxima of the angular intensity distribution are at

$$\theta_1=\pm\frac{\pi}{4}\quad\text{and}\quad\theta_2=\pm\frac{3\pi}{4}$$

for which we have $\sin^2(\theta_{1,2})=\cos^2(\theta_{1,2})=1/2$, and, therefore,

$$\left(\frac{dP}{d\Omega}\right)_{\rm max}=\frac{c}{32\pi}\,k^2\,\left|A_\phi^0\right|\,.$$

(d) The intensity at these values over the average intensity is given by

$$g=\frac{\sin^2(\theta_{1,2})\,\cos^2(\theta_{1,2})}{\int_0^{2\pi}\int_{-1}^{+1}\sin^2(\theta)\,\cos^2(\theta)\,d\cos(\theta)\,d\phi/(4\pi)}\,.$$

$$=\frac{1}{4\int_0^1(1-x^2)\,x^2\,dx}=\frac{1}{4(1/3-1/5)}=\frac{15}{8}\,.$$