Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 3:

4. Magnetism in matter.

(1) For this magnetostatics problem with no free currents the relevant Maxwell equations are

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{H} = 0$$

with the associated boundary conditions (BCs)

$$\hat{r} \cdot \left(\vec{B}_2 - \vec{B}_1\right)\Big|_{r=a,b} = 0\,, \qquad \hat{r} \times \left(\vec{H}_2 - \vec{H}_1\right)\Big|_{r=a,b} = 0\,.$$

(2) Since $\nabla \times \vec{H} = 0$ we have $\vec{H} = -\nabla\Phi$ and thus $0 = \nabla \cdot \vec{B} = \nabla \cdot \left(\mu \vec{H}\right)$. From the BCs we get

$$0 = \hat{r} \cdot \left(\vec{B}_2 - \vec{B}_1\right) = \hat{r} \cdot \left(\mu_2 \vec{H}_2 - \mu_1 \vec{H}_1\right) = -\mu_2 \frac{\partial \Phi_2}{\partial r} + \mu_1 \frac{\partial \Phi_1}{\partial r}$$

and, because of symmetry $\Phi(\vec{x}) = \Phi(r, \theta)$,

$$0 = \hat{r} \times \left(\vec{H}_2 - \vec{H}_1\right) = \hat{r} \times \left(-\nabla\Phi_2 + \nabla\Phi_1\right) = \frac{\hat{r} \times \hat{\theta}}{r} \left(-\frac{\partial \Phi_2}{\partial \theta} + \frac{\partial \Phi_1}{\partial \theta}\right)\,.$$

(3) The solution is dictated by the behavior of the field at infinity:

$$\lim_{r\to\infty} \vec{B}(\vec{r}) = \vec{B}_0 = B_0\, \hat{z} \;\Rightarrow\; \lim_{r\to\infty} \Phi(\vec{r}) = -B_0\, z = -B_0\, r\, \cos\theta = -B_0\, r\, P_1(\cos\theta)\,.$$

Thus, we obtain for the $P_l(\cos\theta)$ Legendre polynomials

$$\Phi_1(\vec{r}) = A\, r^l\, P_l(\cos\theta) \quad \text{for } r < a \quad \text{(no singular contribution)}\,,$$
$$\Phi_2(\vec{r}) = \left(B\, r^l + \frac{C}{r^{l+1}}\right) P_l(\cos\theta) \quad \text{for } a < r < b\,,$$
$$\Phi_3(\vec{r}) = -\delta^{1l}\, B_0\, r\, \cos\theta + \frac{D}{r^{l+1}}\, P_l(\cos\theta) \quad \text{for } r > b\,.$$

Using the BCs we obtain:

$$A\, a^l = B\, a^l + \frac{C}{a^{l+1}} \quad \text{(continuity of } \hat{r} \times \vec{H} \text{ at } r = a)\,,$$
$$l\, A\, a^{l-1} = \mu \left(l\, B - (l+1)\, \frac{C}{a^{l+2}}\right) \quad \text{(continuity of } \hat{r} \cdot \vec{B} \text{ at } r = a)\,,$$
$$B\, b^l + \frac{C}{b^{l+1}} = -\delta^{1l} B_0\, b + \frac{D}{b^{l+1}} \quad \text{(continuity of } \hat{r} \times \vec{H} \text{ at } r = b)\,,$$

$$\mu\left(l\,B\,b^{l-1}-(l+1)\,\frac{C}{b^{l+2}}\right)=$$
$$-\,\delta^{1l}B_0-(l+1)\,\frac{D}{b^{l+2}}\quad\text{(continuity of }\hat{r}\cdot\vec{B}\text{ at }r=b)\,.$$

For $l\geq 2$ a solution is $A=B=C=D=0$ and it is the only solution as long as the determinant of the associated linear system of four equations with four unknowns is not zero. Thus, we keep only $l=1$:

$$A\,a=B\,a+\frac{C}{a^2}\quad\text{(continuity of }\hat{r}\times\vec{H}\text{ at }r=a)\,,$$
$$A=\mu\left(B-\frac{2C}{a^3}\right)\quad\text{(continuity of }\hat{r}\cdot\vec{B}\text{ at }r=a)\,,$$
$$B\,b+\frac{C}{b^2}=-B_0\,b+\frac{D}{b^2}\quad\text{(continuity of }\hat{r}\times\vec{H}\text{ at }r=b)\,,$$
$$\mu\left(B-\frac{2C}{b^3}\right)=-B_0-\frac{2D}{b^3}\quad\text{(continuity of }\hat{r}\cdot\vec{B}\text{ at }r=b)\,.$$

From the BCs at $r=a$ we obtain

$$B=\left(\frac{1+2\mu}{3\mu}\right)A\,,\quad C=\left(\frac{\mu-1}{3\mu}\right)a^3\,A\,,$$

and from the BCs at $r=b$ we get

$$B=-\left(\frac{1+2\mu}{3\mu}\right)B_0+\frac{2\,(\mu-1)}{3\mu\,b^3}\,D\,,\quad C=-\left(\frac{\mu-1}{3\mu}\right)b^3\,B_0+\left(\frac{2+\mu}{3\mu}\right)D\,.$$

Thus, the four unknown constant can all be determined in terms of B_0. Eliminating B and C in favor of A in the last two equations, these become

$$\left(\frac{1+2\mu}{3\mu}\right)A=-\left(\frac{1+2\mu}{3\mu}\right)B_0+\frac{2\,(\mu-1)}{3\mu\,b^3}\,D\,,$$
$$\left(\frac{\mu-1}{3\mu}\right)a^3\,A=-\left(\frac{\mu-1}{3\mu}\right)b^3\,B_0+\left(\frac{2+\mu}{3\mu}\right)D\,,$$

with the solutions

$$A=\frac{-9\,b^3\,\mu\,B_0}{b^3\,(2\mu+1)\,(\mu+2)-2a^3\,(\mu-1)^2}\,,\quad D=\frac{(2\mu+1)\,(\mu-1)\,(b^3-a^3)\,b^3\,B_0}{b^3\,(2\mu+1)\,(\mu+2)-2a^3\,(\mu-1)^2}\,.$$

The constants B and C for the region $a < r < b$ are best expressed in terms of A. That is,

$$B = \left(\frac{1+2\mu}{3\mu}\right) A\,, \quad C = \left(\frac{\mu-1}{3\mu}\right) a^3\, A\,.$$

Note that for $B_0 = 0$ all coefficient become zero.

(4) In the limit $\mu \to \infty$ we obtain $A \sim B_0/\mu$ and, hence, $A = B = C = 0$ (no field in the inner sphere) and

$$D \;\to\; \frac{2\mu^2\,(b^3-a^3)\,b^3\,B_0}{2\mu^2\,(b^3-a^3)} \;=\; b^3\,B_0\,.$$

Thus,

$$\Phi_1(\vec{r}) \;=\; 0 \quad \text{and} \quad \Phi_3(\vec{r}) = \left(-B_0\,r + \frac{b^3}{r^2}\,B_0\right)\cos\theta$$
$$= -B_0\,z + \frac{\vec{m}\cdot\vec{r}}{r^3} \;\;\text{with}\;\; \vec{m} = b^3\,B_0\,\hat{z}\,.$$

We obtain

$$B_3(\vec{r}) = -\nabla\Phi_3(\vec{r}) \;=\; B_0\,\hat{z} - (\vec{m}\cdot\vec{r}\,)\,\nabla\,\frac{1}{r^3} - \frac{1}{r^3}\,\nabla\,(\vec{m}\cdot\vec{r}\,)$$
$$= \vec{B}_0 + \frac{3\,(\vec{m}\cdot\vec{r}\,)\,\hat{r}}{r^4} - \frac{\vec{m}}{r^3} \;=\; \vec{B}_0 + \frac{3\,(\vec{m}\cdot\hat{r}\,)\,\hat{r}-\vec{m}}{r^3}\,,$$

i.e., we find a magnetic dipole correction to the constant magnetic field.

5. Faraday's law of induction in a constant magnetic field.

We use the initial condition $\hat{n} = \hat{y}$ at time $t = 0$, where $\hat{n}$ is the normal to the surface.

(1) For the left-hand side we have to calculate

$$-\frac{1}{c}\,\frac{\Phi_m}{d\,t} \;\;\text{with}\;\; \Phi_m \;=\; \int_S \vec{B}\cdot\hat{n}\,da \;\;\text{and}\;\; \hat{n} = \hat{\phi} = -\sin(\omega\,t)\,\hat{x} + \cos(\omega\,t)\,\hat{y}$$

for the normal to the surface, where we use cylindrical coordinates and $\phi = \omega t$. With $\vec{B} = B_0\,\hat{y}$ we obtain for the left-hand side the result

$$\vec{B}\cdot\hat{n} = B_0\,\cos(\omega\,t) \;\;\text{and}\;\; \Phi_m = B_0\,L^2\,\cos(\omega\,t)\,,$$
$$-\frac{1}{c}\,\frac{d}{dt}\,\Phi_m \;=\; c^{-1}B_0\,L^2\,\omega\,\sin(\omega\,t)\,.$$

On the right-hand side the electric field $\vec{E}$ does not contribute, because we have a constant magnetic field and, hence,

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0 \;\Rightarrow\; \oint_C \vec{E}\cdot d\vec{l} = 0\,,$$

so that the equation becomes

$$\frac{1}{c}\oint_C \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l},$$

where the velocity is that of the line element $d\vec{l}$. With $\rho = \sqrt{x^2+y^2}$

$$\begin{aligned}\vec{v} &= \rho\,\omega\,\hat{\phi} = \rho\,\omega\,\left[-\sin(\omega\,t)\,\hat{x} + \cos(\omega\,t)\,\hat{y}\right]\\ \vec{v}\times\vec{B} &= -\rho\,\omega\,B_0\,\sin(\omega\,t)\,\hat{x}\times\hat{y} \;=\; -\rho\,\omega\,B_0\,\sin(\omega\,t)\,\hat{z}\,.\end{aligned}$$

Therefore, only the line elements in $\hat{z}$ direction contribute, which are at $\rho = L/2$. The integral becomes (note the right-handed orientation of the loop)

$$\begin{aligned}\frac{1}{c}\oint_C \left(\vec{v}\times\vec{B}\right)\cdot d\vec{l} &= -\frac{B_0\,L\,\omega}{2\,c}\,\sin(\omega\,t)\left[\int_L^0 dz - \int_0^L dz\right]\\ &= c^{-1}B_0\,L^2\,\omega\,\sin\omega\,t)\,.\end{aligned}$$

(2) We have

$$\nabla\cdot\vec{v} = \left(\hat{\rho}\,\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\,\frac{\partial}{\partial\phi} + \hat{z}\,\frac{\partial}{\partial z}\right)\cdot\omega\,\rho\,\hat{\phi} = 0 \quad\Rightarrow\quad \frac{1}{c}\int_S \left(\nabla\cdot\vec{v}\right)\,\vec{B}\cdot d\vec{a} = 0\,.$$

Next,

$$\begin{aligned}\left(\vec{B}\cdot\nabla\right)\,\vec{v} = B_0\,\frac{\partial\vec{v}}{\partial y} \;&=\; B_0\,\hat{y}\cdot\left(\hat{\rho}\,\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\,\frac{\partial}{\partial\phi} + \hat{z}\,\frac{\partial}{\partial z}\right)\,\vec{v}\\ &= B_0\,\left(\sin(\omega t)\,\frac{\partial}{\partial\rho} + \frac{\cos(\omega t)}{\rho}\,\frac{\partial}{\partial\phi}\right)\,\omega\,\rho\,\hat{\phi}\\ &=\; B_0\,\omega\,\left(\sin(\omega t)\,\hat{\phi} - \cos(\omega t)\,\hat{\rho}\right)\end{aligned}$$

and, therefore (using $\hat{\phi}\cdot\hat{n} = 1$ and $\hat{\rho}\cdot\hat{n} = 0$),

$$\frac{1}{c}\int_S \left(\vec{B}\cdot\nabla\right)\,\vec{v}\cdot d\vec{a} \;=\; c^{-1}\omega\,B_0\,L^2\,\sin(\omega t)\,.$$

Further

$$\frac{1}{c}\int_S (\vec{v}\cdot\nabla)\,\vec{B}\cdot d\vec{a} \;=\; 0\,,$$

because $\vec{B}$ is constant. Finally,

$$\vec{v}\times\vec{B} = -\rho\,\omega\,B_0\,\sin(\omega t)\,\hat{z}\,,$$

$$\nabla\times\rho\,\omega\,B_0\,\sin(\omega t)\,\hat{z} = -\omega\,B_0\,\sin(\omega t)\,\hat{\rho}\times\hat{z} \;=\; +\omega\,B_0\,\sin(\omega t)\,\hat{\phi} - \omega\,B_0\,\sin(\omega t)\,\hat{\rho}$$

and $\hat{\phi}\cdot\hat{n} = 1$ implies

$$\frac{1}{c}\int_S \nabla\times\left(\vec{v}\times\vec{B}\right)\cdot d\vec{a} \;=\; c^{-1}B_0\,L^2\,\omega\,\sin(\omega\,t)$$

as before. The results are consistent with the vector relation

$$\nabla\times\left(\vec{v}\times\vec{B}\right) \;=\; \left(\nabla\cdot\vec{B}\right)\vec{v} + \left(\vec{B}\cdot\nabla\right)\vec{v} - (\nabla\cdot\vec{v})\,\vec{B} - (\vec{v}\cdot\nabla)\,\vec{B}\,.$$

(3) We have now $\hat{n} = \hat{x}$ and, therefore,

$$\Phi_m \;=\; \int_S \vec{B}\cdot\hat{n}\,da \;=\; B_0\,L^2\,\sin(\omega\,t)\,.$$

Differentiation with respect to the time gives

$$-\frac{1}{c}\,\frac{d}{dt}\,\Phi_m = -c^{-1}B_0\,L^2\,\omega\,\cos(\omega\,t)\,.$$

This is the previous result phase-shifted due to different initial conditions.

6. LCR circuit.

(1) As function of the angular frequency the maximum of the current is given by

$$I_{\max}(\omega) = \frac{\epsilon_{\max}}{Z} = \frac{\epsilon_{\max}}{\sqrt{\left[1/(\omega C) - \omega L\right]^2 + R^2}}\,,$$

which implies for the resonance frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{2\,[H]\,2\,10^{-6}\,[F]}} = \frac{1}{4\pi\,10^{-3}\,[s]} = 79.6\,[Hz]\,.$$

(2) Let $\epsilon_{\max} = 100\,[V]$ and $\omega_{60} = 120\pi\,[Hz]$. Then,

$$X_C \;=\; \frac{1}{\omega_{60} C} = 1326.3\,[V/A]\,, \qquad X_L \;=\; \omega_{60} L \;=\; 754.0\,[V/A]\,,$$
$$I_{\max}(\omega_{60}) = \frac{100\,[V]}{\sqrt{(X_C - X_L)^2 + R^2}} = 0.175\,[A]\,,$$

whereas at resonance frequency we have $I_{\max}(\omega_0) = \epsilon_{\max}/R = 5\,[A]$. This is with 28.6 almost thirty times larger than the maximum for ω_{60}. For the charge maximum we have

$$Q_{\max}(\omega) = \frac{\epsilon_{\max}}{\omega\, Z} = \frac{I_{\max}(\omega)}{\omega}\,.$$

resulting in $Q_{\max}(\omega_{60}) = 0.00046\,[C]$ and $Q_{\max}(\omega_0) = 0.01\,[C]$. The ratio is down to 21.6.

(3) The phase shift at $60\,[Hz]$ is rather small:

$$\tan\delta \;=\; \frac{R}{X_L - X_C} \;=\; -0.0349 \;\Rightarrow\; \delta \;=\; -2^o\,.$$

7. Complex numbers and integration.

(1) $\overline{z} = x - iy$.

(2) $|z| = +\sqrt{x^2+y^2}$.

(3) $\displaystyle \frac{1}{z} = \frac{\overline{z}}{z\,\overline{z}} = \frac{x-iy}{x^2+y^2}\,.$

(4) $z = |z|\,e^{i\phi} = r\,e^{i\phi}$, $\tan(\phi) = y/x$.

(5) In cylindrical coordinates

$$I_n = \oint_C dz\, z^n = \int_0^{2\pi} R\, i\, e^{i\,\phi}\, d\phi\, R^n\, e^{i\,n\,\phi} = i\, R^{n+1} \int_0^{2\pi} d\phi\, e^{i\,(n+1)\,\phi}$$

For $n \neq -1$:

$$I_n = \frac{i\,R^{n+1}}{i\,(n+1)}\;\; e^{i\,(n+1)\,\phi}\Big|_0^{2\pi} = \frac{R^{n+1}}{(n+1)}\;\left(e^{i\,(n+1)\,2\pi} - 1\right) = 0\,.$$

For $n = -1$:

$$I_{-1} = \oint_C \frac{dz}{z} = i\,\int_0^{2\pi} d\phi = 2\pi\, i\,.$$