Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 1:

1. Solenoid.

(1) For a single loop in the x–y plane, centered at the origin, we have derived

$$\vec{A} = A^\phi\,\hat{\phi} \quad \text{with} \quad A^\phi = \frac{I\,R}{c}\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{\sqrt{R^2+r^2-2R\,r\,\sin\theta\,\cos\phi'}}\ .$$

In the cylindrical coordinates A^ϕ reads

$$A^\phi = \frac{I\,R}{c}\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{\sqrt{R^2+z^2+\rho^2-2R\,\rho\,\cos\phi'}}\ .$$

For $\rho \ll R$ the expansion to order ρ/R is

$$\begin{aligned}
A^\phi &= \frac{I\,R}{c}\left[\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{\sqrt{R^2+z^2}} + \int_0^{2\pi}\frac{R\,\rho\,\cos^2\phi'\,d\phi'}{(R^2+z^2)^{3/2}}\right] + O\left(\frac{\rho^2}{R^2}\right) \\
&= \frac{\pi\,I\,R^2\,\rho}{c\,(R^2+z^2)^{3/2}} \quad \text{as} \quad \int_0^{2\pi}\cos^n\phi'\,d\phi' \ = \begin{cases} 0 & \text{for} \quad n=1\,; \\ \pi & \text{for} \quad n=2\,. \end{cases}
\end{aligned}$$

If the loop has its center not at the origin but at z' on the z-axis, this equation becomes

$$A^\phi \ = \ \frac{\pi\,I\,R^2\,\rho}{c\,(R^2+(z-z')^2)^{3/2}}\,.$$

In this approximation the potential of a solenoid of length L with $n+1$ loops is given by

$$\begin{aligned}
A^\phi &= \sum_{i=0}^{n}\frac{\pi\,I\,R^2\,\rho}{c\,(R^2+(z-z'_i)^2)^{3/2}} \\
&= \frac{n}{L}\sum_{i=0}^{n}\frac{L}{n}\,\frac{\pi\,I\,R^2\,\rho}{c\,(R^2+(z-z'_i)^2)^{3/2}} \quad \text{with} \quad z'_i = -\frac{L}{2}+\frac{i\,L}{n}
\end{aligned}$$

For large n we obtain

$$\begin{aligned}
A^\phi &\approx N\int_{-L/2}^{+L/2}\frac{dz'\,\pi\,I\,R^2\,\rho}{c\,(R^2+(z-z')^2)^{3/2}} \\
&= \frac{\pi\,N\,I\,R^2\,\rho}{c}\int_{-L/2-z}^{+L/2-z}\frac{dz''}{c\,(R^2+z''^2)^{3/2}}\,,
\end{aligned}$$

where $N = n/L$ is the number of loops per unit length and we substituted $z' = z'' + z$ in the last step.
The last integral is elementary

$$\int^z \frac{dz''}{(R^2 + z''^2)^{3/2}} \;=\; \frac{z}{R^2\,\sqrt{R^2 + z^2}} + \text{const}\,,$$

implying

$$A^\phi \;=\; \frac{\pi\, N\, I\, \rho}{c} \left[\frac{L/2 - z}{\sqrt{R^2 + (L/2 - z)^2}} - \frac{-L/2 - z}{\sqrt{R^2 + (L/2 + z)^2}} \right] .$$

From this expression we calculate the magnetic field. First, the z component is

$$B^z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\,\left(\rho\, A^\phi\right) \;=\; \frac{2\pi\, N\, I}{c} \left[\frac{L/2 - z}{\sqrt{R^2 + (L/2 - z)^2}} + \frac{L/2 + z}{\sqrt{R^2 + (L/2 + z)^2}} \right] .$$

To expand for R, $z \ll L$ we factor $L/2$ out of the square root and drop the terms of $1/L^2$ in the square root

$$\begin{aligned} B^z &= \frac{4\pi\, N\, I}{c\, L}\, \left[\frac{L/2 - z}{\sqrt{1 - 4z/L + 4z^2/L^2 + 4R^2/L^2}} + \frac{L/2 + z}{\sqrt{1 + 4z/L + 4z^2/L^2 + 4R^2/L^2}} \right] \\ &= \frac{4\pi\, N\, I}{c\, L}\, \left[\frac{L/2 - z}{\sqrt{1 - 4z/L}} + \frac{L/2 + z}{\sqrt{1 + 4z/L}} \right] . \end{aligned}$$

Finally, we use the following approximations to order $1/L$

$$\sqrt{1 \pm 4z/L} = 1 \pm \frac{2z}{L}\,, \qquad \frac{1}{1 \pm 2z/L} = 1 \mp \frac{2z}{L}$$

with the result

$$B^z = \frac{4\pi\, N\, I}{c}\, \left[1 + O\left(\frac{1}{L^2}\right) \right] \approx \frac{4\pi\, N\, I}{c}\, .$$

Similarly we evaluate the B^ρ component

$$B^\rho \;=\; -\frac{\partial A^\phi}{\partial z} \;=\; \frac{\pi\, N\, I\, R^2\, \rho}{c} \left[\frac{-1}{[R^2 + (L/2 + z)^2]^{3/2}} + \frac{1}{[R^2 + (L/2 - z)^2]^{3/2}} \right] .$$

We factor $L/2$ out

$$B^\rho \,=\, \frac{8\pi\, N\, I\, R^2\, \rho}{c} \quad \left[\frac{-1}{L^3\,[1+4z/L+4z^2/L^2+4R^2/L^2]^{3/2}} \right.$$
$$\left. + \,\frac{1}{L^3\,[1+4z/L-4z^2/L^2+4R^2/L^2]^{3/2}}\right]\,.$$

and expand for R, $z \ll L$ using

$$(1 \pm 4z/L)^{3/2} \,=\, 1 \pm 6\,\frac{z}{L}$$

and find

$$\begin{aligned} B^\rho &= \frac{8\pi\, N\, I\, R^2\, \rho}{c}\left[\frac{-1}{L^3\,[1+4z/L]^{3/2}} + \frac{1}{L^3\,[1-4z/L]^{3/2}}\right]\left[1 + O\left(\frac{1}{L^2}\right)\right] \\ &\approx \frac{8\pi\, N\, I\, R^2\, \rho}{c}\left[\frac{12\, z}{L^4}\right] \,=\, \frac{96\,\pi\, N\, I\, R^2\, z\, \rho}{c\, L^4}\,. \end{aligned}$$

(2) For $z = L/2$ we have

$$\begin{aligned} B^\rho &= \frac{\pi\, N\, I\, R^2\, \rho}{c}\left[\frac{-1}{(R^2+L^2)^{3/2}} + \frac{1}{R^3}\right] \\ &= \frac{\pi\, N\, I\, R^2\, \rho}{c}\,\left(\frac{1}{R^3}\right)\,\left[O\left(\frac{1}{L^2}\right) + 1\right] \approx \frac{\pi\, N\, I\, \rho}{c\, R}\,. \end{aligned}$$

and

$$B^z = \frac{2\pi\, N\, I\, R^2}{c}\left[-\frac{-L}{R^2\,\sqrt{R^2+L^2}}\right] = \frac{2\pi\, N\, I}{c}\,\left[1 + O\left(\frac{1}{L^2}\right)\right] \approx \frac{2\pi\, N\, I}{c}\,.$$