Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 8

20. 1D wave in a dispersive medium (E.80).

(a) Due to the e^{-iwt} dependence we get at fixed ω

$$\frac{d^2u}{d^2x} \;=\; -\frac{\omega^2}{v^2}\,u$$

with the fundamental solutions $\exp[\pm i\,k(\omega)\,x]$, $k(w) = \omega/v(\omega)$ so that the general solution is the superposition

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\,\left[A(\omega)\,e^{ik(\omega)x} + B(\omega)\,e^{-ik(\omega)x}\right]\;.$$

With $k(\omega) = n(\omega)\,\omega/c$ the desired results follows.

(b) At $t = 0$ we have

$$u(0,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\,\left[A(\omega) + B(\omega)\right]\,,$$

$$\frac{\partial u(0,t)}{\partial\, x} = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\,\left[\frac{n(\omega)\,\omega}{c}\,A(\omega) - \frac{n(\omega)\,\omega}{c}\,B(\omega)\right]\,.$$

Inverse Fourier transformation:

$$A(\omega) + B(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\,u(0,t)\,,$$

$$i\,\frac{n(\omega)\,\omega}{c}\,\left[A(\omega) - B(\omega)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\,\frac{\partial u(0,t)}{\partial\, x}\,,$$

Addition and subtraction give the final results:

$$A(\omega) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\,\left[u(0,t) - \frac{i\,c}{n(\omega)\,\omega}\,\frac{\partial u(0,t)}{\partial\, x}\right]\,,$$

$$B(\omega) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\,\left[u(0,t) + \frac{i\,c}{n(\omega)\,\omega}\,\frac{\partial u(0,t)}{\partial\, x}\right]\,,$$

21. Energy density and Poynting vector for plane waves (E.83).

(a) Energy and Poynting vector are given by

$$u = \frac{1}{8\pi}\left(\vec{E}^{\,2} + \vec{B}^{\,2}\right), \qquad \vec{S} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right) .$$

We have to insert the physical fields

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\,\vec{x} - \omega t)\,, \qquad \vec{B} = \vec{B}_0 \cos(\vec{k}\,\vec{x} - \omega t)$$

$$\text{with} \quad \omega = c\,k\,, \quad k = |\vec{k}|\,, \quad \vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{k}\,, \quad \vec{E}_0 \perp \vec{k}\ .$$

Using

$$\left(\vec{k}\times\vec{E}_0\right)\cdot\left(\vec{k}\times\vec{E}_0\right) = k^2\,\vec{E}_0^{\,2} - \left(\vec{k}\cdot\vec{E}_0\right)^2 = k^2\,\vec{E}_0^{\,2}$$

The energy density becomes

$$u = \frac{1}{8\pi}\left(\vec{E}_0^{\,2} + \vec{E}_0^{\,2}\right)\cos^2\left(\vec{k}\cdot\vec{x} - \omega t\right) = \frac{1}{4\pi}\,\vec{E}_0^{\,2}\,\cos^2\left(\vec{k}\cdot\vec{x} - \omega t\right)$$

The time average over a period $T = 2\pi/\omega$ is

$$\widehat{u} = \frac{1}{T}\int_0^T u(t)\,dt = \frac{1}{8\pi}\,\vec{E}_0^{\,2}\ .$$

Using

$$\vec{E}_0\times\left(\vec{k}\times\vec{E}_0\right) = -\left(\vec{k}\cdot\vec{E}_0\right)\vec{E}_0 + \vec{k}\left(\vec{E}_0\right)^2 = \vec{k}\left(\vec{E}_0\right)^2$$

we find for the Poynting vector

$$\vec{S} = \frac{c\,\vec{k}}{4\pi\,k}\left(\vec{E}_0\right)^2\cos^2\left(\vec{k}\cdot\vec{x} - \omega t\right)\ .$$

The time average over a period $T = 2\pi/\omega$ is

$$\widehat{\vec{S}} = \frac{1}{T}\int_0^T \vec{S}(t)\,dt = \frac{c\,\vec{k}}{8\pi\,k}\left(\vec{E}_0\right)^2\ .$$

(b) In complex notation we have

$$\vec{E} = \vec{E}_0\,e^{i\vec{k}\cdot\vec{x} - i\omega t} \quad \text{and} \quad \vec{B} = \frac{\vec{k}\times\vec{E}_0}{k}\,e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

and, therefore,

$$\vec{S} = \frac{c}{8\pi}\,\vec{E}\times\overline{\vec{B}} = \frac{c}{8\pi}\,\vec{E}_0 \times \left(\frac{\vec{k}\times\vec{E}_0}{k}\right) = \frac{c\,\vec{k}}{8\pi\,k}\,\left(\vec{E}_0\right)^2$$

which agrees with the time average.
(c) Assuming perfect absorption, we want to calculate the radiation pressure on a surface perpendicular to $\vec{k}$. The average momentum density is

$$\vec{p} = \frac{1}{c^2}\,\vec{S} = \frac{\vec{k}}{8\pi\,k\,c}\,\left(\vec{E}_0\right)^2 ,$$

which gives for the radiation pressure (total momentum transfer per time unit times unit area)

$$|\vec{p}|\,(c\,\triangle t)\,\triangle a/(\triangle t\,\triangle a) = |\vec{p}|\,c = \frac{1}{8\pi}\,\left|\vec{E}_0\right|^2 .$$

22. Reflection and transmission of a circularly polarized wave (E.89).

The incident, circularly polarized wave is

$$\vec{E} = E_0\,(\hat{\epsilon}_1 \pm i\,\hat{\epsilon}_2)\,e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

where the directions $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ correspond to perpendicular and parallel polarized light, respectively.

For the transmitted wave we choose (instead of linearly polarized components) the convention

$$\vec{E}' = \left(E_1'\,\hat{\epsilon}_1 \pm i\,E_2'\,\hat{\epsilon}_2^{\,'}\right)\,e^{i\vec{k}'\cdot\vec{x}-i\omega t} ,$$

so that E_0, E_1' and E_2' are in phase. For the perpendicularly polarized amplitude we have

$$\frac{E_1'}{E_0} = \frac{2n\,\cos(\theta_i)}{n\cos(\theta_i)+n'\,\cos(\theta_t)}$$

and for the parallel polarized amplitude

$$\frac{E_2'}{E_0} = \frac{2n\,\cos(\theta_i)}{n'\cos(\theta_i)+n\,\cos(\theta_t)}$$

For $\theta_i < \theta_0$, where θ_0 is the angle of total reflection, the phases stay unchanged. Therefore, the axes of the ellipse are along $\hat{\epsilon}_1$ and $\hat{\epsilon}_2^{'}$. The ratio of the axes is

$$\frac{E_2'}{E_1'} = \frac{n\,\cos(\theta_i) + n'\,\cos(\theta_t)}{n'\,\cos(\theta_i) + n\,\cos(\theta_t)} \quad \text{with} \quad \frac{\sin(\theta_t)}{\sin(\theta_i)} = \frac{n}{n'}\ .$$

Linear polarization,

$$E_1' \;=\; 0 \quad \text{or} \quad E_2' \;=\; 0\,,$$

are both impossible as they imply $\cos(\theta_i) = 0$.
For the reflected wave we choose the convention

$$\vec{E}'' = \left(E_1''\,\hat{\epsilon}_1 \pm i\,E_2''\,\hat{\epsilon}_2^{''}\right)\,e^{i\vec{k}''\cdot\vec{x}-i\omega t}\ .$$

For the perpendicular polarized amplitude we have

$$\frac{E_1''}{E_0} = \frac{n\,\cos(\theta_i) - n'\,\cos(\theta_t)}{n\,\cos(\theta_i) + n'\,\cos(\theta_t)}$$

and for the parallel polarized amplitude

$$\frac{E_2''}{E_0} = \frac{n'\,\cos(\theta_i) - n\,\cos(\theta_t)}{n'\,\cos(\theta_i) + n\,\cos(\theta_t)}$$

As the phases are not changed, the axes of the ellipse are along $\hat{\epsilon}_1$ and $\hat{\epsilon}_2^{''}$. The ratio of the axes is

$$\frac{E_2''}{E_1''} = \frac{(n'\,\cos\theta_i - n\,\cos\theta_t)\,(n\,\cos\theta_i + n'\,\cos\theta_t)}{(n'\,\cos\theta_i + n\,\cos\theta_t)\,(n\,\cos\theta_i - n'\,\cos\theta_t)}$$

Linearly polarized light: For

$$E_2'' \;=\; 0 \quad \text{we have} \quad n'\,\cos\theta_i - n\,\cos\theta_t = 0$$

with the non-trivial solution

$$\tan(\theta_i) = \frac{n'}{n} \quad (\text{Brewster's angle})\ .$$

For

$$E_1'' \;=\; 0 \quad \text{we have} \quad n\,\cos\theta_i - n'\,\cos\theta_t = 0$$

with only the trivial solution $n' = n$, $\theta_t = \theta_i$ for which the wave remains circularly polarized as there is no reflection and $E_1' = E_2' = E_0$.