Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 5:

11. Wave equation with point source.

The wave is obtained by integrating the source with the retarded Green function (the homogeneous solution is zero as the wave disappears for times $t < 0$):

$$\begin{aligned}\psi_r(x^0,\vec{x}) &= \int dy^0\, d^3y\, G_r(x-y)\,\theta(x^0)\,\delta(\vec{y}\,)\\ &= \int dy^0\, d^3y\,\frac{\delta(x^0-y^0-|\vec{x}-\vec{y}\,|)}{4\pi\,|\vec{x}-\vec{y}\,|}\,\theta(y^0)\,\delta(\vec{y}\,)\\ &= \int_0^\infty dy^0\,\frac{\delta(x^0-y^0-|\vec{x}|)}{4\pi\,|\vec{x}|} \;=\; \begin{cases}1/(4\pi\,|\vec{x}|) \;\text{ for }\; |\vec{x}|<x^0=c\,t\,;\\ 0 \;\text{ for }\; |\vec{x}|>x^0=c\,t\,.\end{cases}\end{aligned}$$

Using the advanced Green function one gets

$$\psi_a(x^0,\vec{x}) = \begin{cases}1/(4\pi\,|\vec{x}|) \;\text{ for }\; |\vec{x}|<-x^0=-c\,t\,;\\ 0 \;\text{ for }\; |\vec{x}|>-x^0=-c\,t\,.\end{cases}$$

12. Covariant retarded Green function.

With $\tau = x^0 - y^0$ the zeros of $\delta((x-y)^2)$ are at

$$\tau^0 = \pm\sqrt{(\vec{x}-\vec{y}\,)^2} = \pm|\vec{x}-\vec{y}\,|\,.$$

Let now

$$f(\tau) = \tau^2 - (\vec{x}-\vec{y}\,)^2\;,$$

so that

$$\left.\frac{df}{d\tau}\right|_{\tau=\tau^0} = f'(\tau^0) = 2\,\tau^0$$

holds. Therefore,

$$\begin{aligned}\delta((x-y)^2) &= \frac{1}{2\,|f'(\tau^0)|}\left[\delta(\tau-|\vec{x}-\vec{y}\,|)+\delta(\tau+|\vec{x}-\vec{y}\,|)\right]\\ &= \frac{1}{2\,|\vec{x}-\vec{y}\,|}\left[\delta(x^0-y^0-|\vec{x}-\vec{y}\,|)+\delta(x^0-y^0+|\vec{x}-\vec{y}\,|)\right]\end{aligned}$$

and the identity of the two expressions for the Green function follows:

$$\frac{1}{2\pi}\,\theta(x^0-y^0)\,\delta((x-y)^2) = \frac{\delta(x^0-y^0-|\vec{x}-\vec{y}\,|)}{4\pi\,|\vec{x}-\vec{y}\,|}\,.$$

13. Four-potential of a moving point particle.

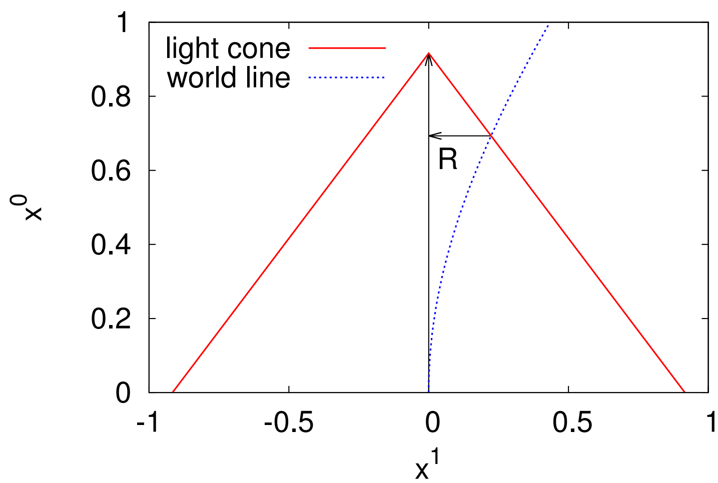

Fig. 0.1 Backward light cone and worldline in Minkowski space.

(1) The distance of the point particle from the origin is given by

$$r(t) = \int_0^t \beta(t')\,dt' \;=\; \int_0^t \tanh(t)'\,dt' = \int_0^{\cosh(t)} \frac{d\cosh(t')}{\cosh(t')} = \ln\left[\cosh(t)\right]\,.$$

(2) With $t_1 = 0.6931472$ the speed is $\beta(t_1) = 0.6$ and the position is $r(t_1) = 0.2231436 = R$.

(3) The position $r(t_1) = 0.2231436$ of the point particle is observed at $\vec{x} = 0$ when the time is $x^0 = t_1 + r(t_1) = 0.9162908$.

(4) We find the corresponding potentials from the equations

$$\Phi\left(x^0,\vec{x}\right) = \left[\frac{q}{\left(1-\vec{\beta}\cdot\hat{n}\right)\,R}\right]_{\text{ret}}\,,\quad \vec{A}\left(x^0,\vec{x}\right) = \left[\frac{q\,\vec{\beta}}{\left(1-\vec{\beta}\cdot\hat{n}\right)\,R}\right]_{\text{ret}}\,.$$

We have $\vec{\beta} = \beta(t_1)\,\hat{x}^1 = 0.6\,\hat{x}^1$, $R = 0.2231436$ and $\vec{R} = -R\,\hat{x}^1$ because the point charge moves away from the origin. Therefore,

$$\Phi = \frac{q_e}{(1+0.6)\,R} \;=\; \frac{q_e}{0.3570297} \;=\; 2.800888\,q_e\,,$$

$$\vec{A} = \frac{0.6\,q_e\,\hat{x}^1}{(1+0.6)\,R} \;=\; \frac{0.6\,q_e\,\hat{x}^1}{0.3570297} \;=\; 1.680533\,\hat{x}^1\,q_e\,.$$