Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 9

23. Wave with a finite extension (E.84).

We choose the wave vector in z-direction and make the ansatz

$$\vec{E}(t,\vec{x}) = [E_0(x,y)\,(\hat{x} \pm i\hat{y}) + F(x,y)\,\hat{z}]\; e^{ikz-i\omega t}$$

with real functions E_0 and F. Then

$$\nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_0}{\partial x} \pm i\,\frac{\partial E_0}{\partial y} + i\,k\,F(x,y) = 0$$
$$\Leftrightarrow F(x,y) = \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\,\frac{\partial E_0}{\partial y}\right) .$$

Next we use the Maxwell equation

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

together with the assumption that $F(x,y)$ is a slowly varying function of x and y (i.e., $\partial F/\partial x \approx 0$ and $\partial F/\partial y \approx 0$) to find

$$\begin{aligned}
\frac{i\omega}{c}\,B^x &= \left(\frac{\partial F}{\partial y} \pm k\,E_0\right) e^{ikz-i\omega t} \;\approx\; \pm k\,E_0\,e^{ikz-i\omega t} ,\\
\frac{i\omega}{c}\,B^y &= \left(i\,k\,E_0 - \frac{\partial F}{\partial x}\right) e^{ikz-i\omega t} \;\approx\; i\,k\,E_0\,e^{ikz-i\omega t} ,\\
\frac{i\omega}{c}\,B^z &= \left(\pm i\,\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right) e^{ikz-i\omega t} \;=\; \pm k\,F(x,y)\,e^{ikz-i\omega t} .
\end{aligned}$$

Using $\omega/c = k$ (for vacuum) gives

$$\vec{B} \approx \mp i\,\vec{E}\ .$$

The other Maxwell equations are consistent with these results. Example:

$$E_0(x,y) = \frac{1}{\cosh(\epsilon\rho)}\,, \quad \epsilon \text{ small}\,, \quad \rho = \sqrt{x^2+y^2}\,.$$

24. Dispersion figure (E.90).

The plot is that of Lecture Notes, Figure 5.2.

25. Faraday effect (birefringent light) (E.91).

We choose the magnetic field in z-direction, $\vec{B}_0 = B_0 \hat{z}$. Then

$$m\,\ddot{x} + m\,\omega_0^2\,x = q_e\,E^x + \frac{q_e}{c}\,\dot{y}\,B_0\,,$$
$$m\,\ddot{y} + m\,\omega_0^2\,y = q_e\,E^y - \frac{q_e}{c}\,\dot{x}\,B_0\,.$$

These equations separate in circular coordinates

$$\hat{\epsilon}_\pm = \hat{x}_\pm = \frac{1}{\sqrt{2}}\,(\hat{x} \pm i\,\hat{y})\,, \quad x_\pm = \frac{1}{\sqrt{2}}\,(x \mp i\,y)\,, \quad E_\pm = \frac{1}{\sqrt{2}}\,(E^x \mp i\,E^y)\,.$$

With the ansatz $E_\pm = E_\pm^0\,\exp(-i\,\omega\,t)$, $x_\pm = x_\pm^0\,\exp(-i\,\omega\,t)$ we get

$$m\,\ddot{x}_\pm + m\,\omega_0^2\,x_\pm = q_e\,E_\pm \pm i\,\frac{q_e}{c}\,\dot{x}_\pm\,B_0\,,$$

$$-m\,\omega^2\,x_\pm^0 + m\,\omega_0^2\,x_\pm^0 = q_e\,E_\pm^0 \mp \omega\,\frac{q_e}{c}\,x_\pm^0\,B_0\,.$$

$$x_\pm^0 = \frac{q_e\,E_\pm^0}{m\,(\omega_0^2 - \omega^2) \pm \omega\,(q_e/c)\,B_0}$$

from which we find the polarizations $P_\pm = N\,q_e\,x_\pm^0$. This gives the indices of refraction for the two circular polarized components:

$$\epsilon_\pm = 1 + \frac{4\pi\,N\,q_e^2}{m\,(\omega_0^2 - \omega^2) \mp \omega\,(q_e/c)\,B_0}\,, \quad k_\pm = \frac{\omega}{c}\,\sqrt{\epsilon_\pm} = \frac{\omega}{c}\,n_\pm\ .$$

The propagation of the two components of circular polarized light in z-direction is given by

$$E_\pm = E_\pm^0\,e^{i\,k_\pm\,z - i\,\omega\,t}\ .$$

So we find

$$\frac{E_-}{E_+} = \frac{E_-^0}{E_+^0}\,e^{i\,\alpha} \quad \text{with} \quad \alpha = (k_- - k_+)\,z$$

which corresponds as shown after equation (5.50) of the lecture notes to a rotation of the axis by

$$\gamma = \frac{\alpha}{2} = (k_- - k_+)\,\frac{l}{2}$$

when the light travels the distance $z = l$ along the z-axis. The wave stays linearly polarized, while the direction of $\vec{E}$ changes when the wave penetrates into the medium:

$$\vec{E} \ = \ (E_1\,\hat{\epsilon}_1' + E_2\hat{\epsilon}_2')\ e^{ikz-i\omega t}$$

with the E_1, E_2 coefficients and k of the incoming wave and

$$\hat{\epsilon}_1' \ = \hat{\epsilon}_1\,\cos(\gamma) + \hat{\epsilon}_2\,\sin(\gamma)\,,$$

$$\hat{\epsilon}_2' \ = -\hat{\epsilon}_1\,\sin(\gamma) + \hat{\epsilon}_2\,\cos(\gamma)\,.$$

26. Principal value integral (E.93).

Due to the fall-off behavior of the function $f(z)$ a half-circle of radius R about 0 in the upper half-plane does not contribute in the $R \to \infty$ limit:

$$\lim_{R\to\infty}\left|\int_{C_{R+}} dz\,f(z)\right| = \lim_{R\to\infty}\left|\int_0^{2\pi} i\,R\,e^{i\phi}\,f(z)\,d\phi\right|$$
$$\le \lim_{R\to\infty}\int_0^{2\pi}\left|i\,R\,e^{i\phi}\,f(z)\right|\,d\phi \sim \lim_{R\to\infty}\int_0^{2\pi}\frac{d\phi}{R} = \lim_{R\to\infty}\frac{2\pi}{R} = 0\,.$$

(1) For the contour of the first figure we obtain by the Cauchy integral theorem

$$\frac{1}{2\pi i}\lim_{\eta\to 0}\int_{C_1}\frac{f(z)\,dz}{z-\omega_R-i\eta} \ = \ \lim_{\eta\to 0} f(\omega_R+i\eta) \ = \ f(\omega_R)\,.$$

(2) For the contour of the second figure we obtain by the Cauchy integral theorem

$$\frac{1}{2\pi i}\lim_{\eta\to 0}\int_{C_2}\frac{f(z)\,dz}{z-\omega_r-i\eta} \ = \ 0\,.$$

(3) We denote the half-circle of the first figure by $C_+(\omega_R+i\eta,\eta)$. To evaluate the integral

$$I_{C_+(\omega_R+i\eta,\eta)} \ = \ \frac{1}{2\pi\,i}\,\lim_{\eta\to 0}\int_{C_+(\omega_R+i\eta,\eta)}\frac{f(z)\,dz}{z-\omega_R-i\eta}$$

we use the Taylor expansion

$$f(z) \ = \ f(\omega_r+i\eta) + f'(\omega_r+i\eta)\,(z-\omega_r-i\eta) + \dots$$

and note that for z on the half-circle $|z-\omega_r-i\eta|=\eta$ holds. Therefore, the integral becomes

$$\begin{aligned} I_{C_+(\omega_R+i\eta,\eta)} &= \frac{1}{2\pi\, i}\,\lim_{\eta\to 0} f(\omega_r+i\eta)\int_{C_+(\omega_R+i\eta,\eta)}\frac{dz}{z-\omega_R-i\eta} \\ &+\frac{1}{2\pi\, i}\,\lim_{\eta\to 0}\mathcal{O}(\eta)\int_{C_+(\omega_R+i\eta,\eta)}\frac{dz}{z-\omega_R-i\eta} \end{aligned}$$

where by explicit integration (compare a previous exercise) the integrals for the half-circle $C_+(\omega_R+i\eta,\eta)$ are now $\pi\, i$, so that we find

$$\frac{1}{2\pi\, i}\,\lim_{\eta\to 0}\int_{C_+(\omega_R+i\eta,\eta)}\frac{f(z)\,dz}{z-\omega_R-i\eta} \;=\; \frac{1}{2}\,f(\omega_R)\,.$$

Similarly, we denote the half-circle of the second figure by $C_-(\omega_R+i\eta,\eta)$ and find

$$\frac{1}{2\pi\, i}\,\lim_{\eta\to 0}\int_{C_-(\omega_R+i\eta,\eta)}\frac{f(z)\,dz}{z-\omega_R+i\eta} = -\frac{1}{2}\,f(\omega_R)\,,$$

because the integration is now in clockwise instead of anti-clockwise direction.

For illustration, let us consider a half-circle about $z=0$. Then,

$$\begin{aligned} I_{C_+(0,\eta)} &= \frac{1}{2\pi\, i}\,\lim_{\eta\to 0}\int_{C_+(0,\eta)}\frac{f(z)\,dz}{z} \\ &= \frac{1}{2\pi\, i}\,\lim_{\eta\to 0}\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\int_{C_+(0,\eta)}\frac{z^n\,dz}{z} \end{aligned}$$

and for the last integral we find

$$\int_{C_+(0,\eta)}\frac{z^n\,dz}{z} \;=\; \int_0^{\pi}\eta\, i\, e^{i\phi}\,d\phi\,\eta^{n-1}\,e^{i\,(n-1)\,\phi} \;=\; i\eta^n\int_0^{\pi}e^{i\,n\,\phi}\,d\phi\,.$$

For $n=0$ the result is $i\,\pi$, while for $n\neq 0$ we find

$$i\eta^n\int_0^{\pi}e^{i\,n\,\phi}\,d\phi \;=\; \frac{i\,\eta^n}{i\,n}\;e^{i\,n\,\phi}\Big|_0^{\pi} \;=\; \begin{cases}0 \text{ for } n \text{ even}\\ 2\eta^n/n \text{ for } n \text{ odd}\,.\end{cases}$$

Therefore, only $n=0$ survives the $\eta\to 0$ limit and

$$I_{C_+(0,\eta)} = \frac{1}{2\pi\, i}\,\lim_{\eta\to 0}\int_{C_+(0,\eta)}\frac{f(z)\,dz}{z} \;=\; \frac{1}{2}\,f(0)\,.$$