Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 6:

14. Charge conjugation, parity and time reversal in electrodynamics.

(1) Using the given values $(-,+,+)$ for ρ as input, our (C,P,T) results are obtained as follows:

$$\begin{aligned}
1.\ A^0 \sim \rho &\ \Rightarrow\ (-,+,+)\,,\\
2.\ \vec{J} \sim \vec{v}\,\rho &\ \Rightarrow\ (-,-,-)\,,\\
3.\ \vec{A} \sim \vec{J} &\ \Rightarrow\ (-,-,-)\,,\\
4.\ E^i = F^{i0} = \partial^i A^0 - \partial^0 A^i &\ \Rightarrow\ (-,-,+)\,,\\
5.\ B^k \sim F^{ij} = \partial^i A^j - \partial^j A^i &\ \Rightarrow\ (-,+,-)\,,\\
6.\ \vec{p} = \int \vec{x'}\,\rho(\vec{x'})\,d^3x' &\ \Rightarrow\ (-,-,+)\,,\\
7.\ \vec{m} = \frac{1}{2c}\int \vec{x'}\times\vec{J}(\vec{x'})\,d^3x' &\ \Rightarrow\ (-,+,-)\,.
\end{aligned}$$

(2) From the explicit matrix forms we find

$$F^{\alpha\beta}\,F_{\alpha\beta} = -2\,\vec{E}^{\,2} + 2\,\vec{B}^{\,2} = -{}^*F^{\alpha\beta}\ {}^*F_{\alpha\beta} \quad \text{and} \quad {}^*F^{\alpha\beta}\,F_{\alpha\beta} = -4\,\vec{E}\cdot\vec{B}\,.$$

Using the (C,P,T) results above, we find $(+,+,+)$ for $F^{\alpha\beta}\,F_{\alpha\beta}$ and ${}^*F^{\alpha\beta}\ {}^*F_{\alpha\beta}$ (proper scalars), and $(+,-,-)$ for ${}^*F^{\alpha\beta}\,F_{\alpha\beta}$ (pseudoscalar).

15. Proca Lagrangian.

(1) From the calculation of the script/lecture we have

$$\partial_\alpha\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_\alpha A_\beta} = -\frac{1}{4\pi}\,\partial_\alpha\,F^{\alpha\beta}\ .$$

Now

$$\frac{\partial\,\mathcal{L}}{\partial\,A_\beta} = \frac{m^2}{8\pi}\left(g^\beta{}_\alpha A^\alpha + A_\alpha g^{\beta\alpha} - \frac{1}{c}J_\alpha g^{\beta\alpha}\right) = \frac{m^2}{4\pi}A^\beta - \frac{1}{c}J_\beta$$

and the Euler-Lagrange equation

$$\partial_\alpha\,\frac{\partial\,\mathcal{L}}{\partial\,\partial_\alpha A_\beta} = \frac{\partial\,\mathcal{L}}{\partial\,A_\beta} \quad \text{reads} \quad \partial_\alpha\,F^{\alpha\beta} + m^2 A^\beta = \frac{4\pi}{c}J^\beta\,.$$

(2) With the Lorentz gauge condition $\partial_\alpha A^\alpha = 0$ we have

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \partial_\alpha \partial^\alpha A^\beta + 0$$

so that the Euler-Lagrange equation becomes

$$\partial_\alpha \partial^\alpha A^\beta + m^2 A^\beta = \frac{4\pi}{c} J^\beta \, .$$

(3) The Proca Euler-Lagrange equation is not gauge invariant, but transforms under the gauge transformation $A^\beta \to A^\beta + \partial^\beta \Lambda$ into

$$\partial_\alpha F^{\alpha\beta} + m^2 A^\beta + m^2 \partial^\beta \Lambda = \frac{4\pi}{c} J^\beta \, .$$

(4) Nevertheless the Lorentz gauge condition should hold: Taking the ∂_β derivative of the Euler-Lagrange equation we find

$$\partial_\beta \partial_\alpha F^{\alpha\beta} + m^2 \, \partial_\beta A^\beta = \frac{4\pi}{c} \, \partial_\beta J^\beta \, .$$

On the left-hand side $\partial_\beta \partial_\alpha F^{\alpha\beta}$ is zero due to the usual argument (contraction of a symmetric with an antisymmetric tensor). On the right-hand side we have $\partial_\beta J^\beta = 0$ due to charge conservation. So we end up with

$$\partial_\beta A^\beta = 0 \; .$$

16. Electromagnetic energy-momentum tensor.

(1) We have

$$\begin{aligned} T^{00} &= -\frac{1}{4\pi} \left(g^{0\mu} \, F_{\mu\lambda} \, \partial^0 \, A^\lambda - \frac{1}{4} \, g^{00} \, F_{\mu\lambda} \, F^{\mu\lambda} \right) \\ &= -\frac{1}{4\pi} \, F^0{}_\lambda \, \partial^0 \, A^\lambda + \frac{1}{8\pi} \left(\vec{B}^{\,2} - \vec{E}^{\,2} \right) \end{aligned}$$

and now

$$\begin{aligned} -F^0{}_\lambda \, \partial^0 \, A^\lambda &= + \sum_i F^{0i} \, \partial^0 \, A^i \; = \; - \sum_i \left[E^i \, \left(\partial^0 \, A^i - \partial^i \, A^0 \right) + E^i \, \partial^i \, A^0 \right] \\ &= \vec{E}^{\,2} + \nabla \cdot \left(\vec{E} \, \Phi \right) \; . \end{aligned}$$

In the last step the relations $\Phi = A^0$, $\nabla \cdot \vec{E} = 0$ and $\nabla = (-\partial^i)$ were used. Putting the above equations together, we have the desired result

$$T^{00} = \frac{1}{8\pi} \left(\vec{E}^{\,2} + \vec{B}^{\,2} \right) + \frac{1}{4\pi} \, \nabla \cdot \left(\Phi \vec{E} \right) \; .$$

(2) We have

$$\Theta^{00} = \frac{1}{4\pi}\, F^0_{\ \lambda}\, F^{\lambda 0} + \frac{1}{16\pi}\, F_{\mu\lambda}\, F^{\mu\lambda}\ .$$

The first term can be re-written as

$$\frac{1}{4\pi}\sum_{i=1}^{3} F^0_{\ i} F^{i0} = \frac{1}{4\pi}\sum_{i=1}^{3} F^{0i} F^{0i} = \frac{1}{4\pi}\,\vec{E}^{\,2}$$

and the second term is known to be $(\vec{B}^{\,2} - \vec{E}^{\,2})/(8\pi)$ from our previous problem. So the result is

$$\Theta^{00} = \frac{1}{8\pi}\left(\vec{E}^{\,2} + \vec{B}^{\,2}\right)$$

Next, as $g^{0i} = 0$ we have

$$\Theta^{0i} = \frac{1}{4\pi} F^{0\lambda} F_\lambda^{\ i} = -\frac{1}{4\pi}\sum_{j\neq i} F^{0j} F^{ji}$$

$i = 3:$

$$\Theta^{03} = -\frac{1}{4\pi}\left(F^{01}F^{13} + F^{02}F^{23}\right) = \frac{1}{4\pi}\left(E^1B^2 - E^2B^1\right) = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^3 .$$

Repeating this evaluation for $i = 1, 2$, we find for all components

$$\Theta^{0i} = \frac{1}{4\pi} F^{0\lambda} F_\lambda^{\ i} = \frac{1}{4\pi}\left(\vec{E}\times\vec{B}\right)^i\ .$$

For Θ^{ij}, $i \neq j$ we find

$$\Theta^{ij} = \frac{1}{4\pi} F^{i\lambda} F_\lambda^{\ j} = \frac{1}{4\pi}\left(F^{i0}F^{0j} - \sum_{k\neq i,j} F^{ik}F^{kj}\right) = -\frac{1}{4\pi}\left(E^iE^j + B^iB^j\right)\ .$$

For Θ^{ii} with **no** summation over i we have

$$\Theta^{ii} = \frac{1}{4\pi}\left(F^{i0}F^{0i} - \sum_{k\neq i} F^{ik}F^{ki} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) =$$

$$\frac{1}{4\pi}\left[-E^iE^i + \vec{B}^{\,2} - B^iB^i - \frac{1}{2}\left(\vec{B}^{\,2} - \vec{E}^{\,2}\right)\right] =$$

$$-\frac{1}{4\pi}\left[E^iE^i + B^iB^i - \frac{1}{2}\left(\vec{E}^{\,2} + \vec{B}^{\,2}\right)\right]\ .$$

All Θ^{ij} together:

$$\Theta^{ij} = -\frac{1}{4\pi}\left[E^iE^j + B^iB^j + \frac{1}{2}\,g^{ij}\,\left(\vec{E}^{\,2} + \vec{B}^{\,2}\right)\right]\ .$$