Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 11

30. Skin depth (E.95).

(1) We define a good conductor by the validity of the approximation $\sigma^c >> \omega\epsilon^c$ so that we have

$$\vec{E}^c = \frac{c}{4\pi\,\sigma^c}\,\nabla\times\vec{H}^c\,.$$

Using the homogeneous Maxwell equation for the electric field together with the $e^{-i\,\omega\,t}$ time dependence of the magnetic field one finds

$$\nabla\times\vec{E}^c = -\,\frac{\mu^c}{c}\,\frac{\partial\vec{H}^c}{\partial t} = \frac{i\,\omega\mu^c}{c}\,\vec{H}^c \quad\text{or}\quad \vec{H}^c = -\frac{i\,c}{\mu^c\,\omega}\,\nabla\times\vec{E}^c\,.$$

We assume smooth variations of the fields parallel to the interface and approximate the nabla operator by a derivative perpendicular to the interface

$$\nabla \;\approx\; \hat{n}\,\frac{\partial}{\partial n}\,.$$

Here $\hat{n}$ is the unit normal vector pointing *into* the conductor and n is the corresponding coordinate. In this approximation

$$\vec{E}^c = \frac{c}{4\pi\,\sigma^c}\,\hat{n}\times\frac{\partial\vec{H}^c}{\partial n} = \frac{c}{4\pi\,\sigma^c}\,\hat{n}\times\frac{\partial\vec{H}^c_\parallel}{\partial n} = \vec{E}^c_\parallel\,,$$

because $\hat{n}\times\vec{H}^c_\perp = 0$ holds. Further,

$$\begin{aligned}\vec{H}^c \approx \vec{H}^c_\parallel &= -\frac{i\,c}{\omega\,\mu^c}\,\hat{n}\times\frac{\partial\vec{E}^c_\parallel}{\partial n} \;=\; -\frac{i\,c^2}{4\pi\,\omega\,\mu^c\,\sigma^c}\,\hat{n}\times\left(\hat{n}\times\frac{\partial^2\vec{H}^c_\parallel}{\partial\,n^2}\right)\\ &= \frac{i\,c^2}{4\pi\,\omega\,\mu^c\,\sigma^c}\,\frac{\partial^2\vec{H}^c_\parallel}{\partial\,n^2}\,,\end{aligned} \tag{0.1}$$

where the previous relation between $\vec{E}^c_\parallel$ and $\vec{H}^c_\parallel$ has been used. The resulting differential equation is elementary:

$$\frac{\partial^2\vec{H}^c_\parallel}{\partial n^2} = -\frac{2\,i}{\delta^2}\,\vec{H}^c_\parallel \quad\text{where}\quad \delta = \frac{c}{\sqrt{4\pi}}\sqrt{\frac{2}{\omega\,\mu^c\,\sigma^c}}$$

is the skin depth. Its interpretation follows from the solution

$$\vec{H}^c_\parallel = \vec{H}^0_\parallel \, e^{-n/\delta} \, e^{+i\,n/\delta} \, .$$

Proof of this equation: The ansatz $\exp(\alpha\, n)$ gives $\alpha^2 = -2i/\delta^2$, i.e., $\alpha = \pm\sqrt{2}\,\sqrt{-i}/\delta$. The square root of $-i$ is taken by writing $-i = \exp(-i\,\pi/2) \Rightarrow \sqrt{-i} = \exp(-i\,\pi/4) = (1-i)/\sqrt{2}$. The result follows after discarding the plus sign solution as unphysical. It remains $\alpha = -(1-i)/\delta$.

(2) (2) Using the table of appendix B,

$$\mu = 10^3\,, \quad \omega = 2\pi\,10^{10}\,[rad/s]\,, \quad \sigma^c = 1/\rho^c = \frac{c_f^2\,10^9}{20}\,\left[\frac{statampere}{statvolt\,m}\right]\,,$$

$$\text{so that} \quad \delta = \frac{c}{\sqrt{4\pi}}\,\sqrt{\frac{2}{\mu\,\omega\,\sigma^c}} = 0.0712\,[cm]\ .$$

However, with $\epsilon = 3$ we find

$$\frac{\epsilon\,\omega}{c} \approx 629 > \frac{4\pi}{c\,\rho^c} \approx 19$$

so that our approximation for the skin depth is invalid. This comes because $\rho^2 = 20\,[\Omega\,m]$ is an unrealistically large resistivity. For instance iron has a resistivity $\rho^c = 10^{-7}\,[\Omega\,m]$. With this we obtain

$$\delta = 0.0523\,[\mu m]\,,$$

and the approximation is valid because

$$\frac{\epsilon\,\omega}{c} \approx 629 \ll \frac{4\pi}{c\,\rho^c} \approx 3.5\times 10^9$$

holds.

31. TM waves in a rectangular wave guide (E.96).

(1)

$$(\nabla_t^2 + \gamma^2)\,E^z = 0\,, \qquad E^z|_S = 0\,,$$

$$E^z = E^0\,\sin\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{n\pi y}{b}\right)\,e^{i\,k\,z}\,e^{-i\,\omega\,t}\,,$$

$$\gamma_{n\,m}^2 = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right] , \quad m = 1, 2, \ldots , \quad n = 1, 2, \ldots ,$$

(2)

$$\gamma_{1\,1}^2 = \pi^2 \left(a^{-2} + b^{-2} \right) , \qquad \omega_{1\,1} = \frac{c\,\pi}{\sqrt{\mu\,\epsilon}} \, \sqrt{a^{-2} + b^{-2}}$$

(3)

$$\vec{E}_t = \frac{i\,k}{\gamma^2} \nabla_t \, E^z , \qquad \text{lowest mode :}$$

$$E^x = E_0 \, \frac{i\,k}{\pi\,a\,(a^{-2} + b^{-2})} \, \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \, e^{i\,k\,z} \, e^{-i\,\omega\,t} ,$$
$$E^y = E_0 \, \frac{i\,k}{\pi\,b\,(a^{-2} + b^{-2})} \, \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \, e^{i\,k\,z} \, e^{-i\,\omega\,t} .$$

(4)

$$\vec{H}_t = \frac{\epsilon\,\omega}{c\,k} \, \hat{z} \times \vec{E}_t , \qquad \text{lowest mode :}$$

$$H^x = -E_0 \, \frac{i}{b\,(a^{-2} + b^{-2})} \sqrt{\frac{\epsilon}{\mu}} \, \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) \, e^{i\,k\,z} \, e^{-i\,\omega\,t} ,$$
$$H^y = +E_0 \, \frac{i}{a\,(a^{-2} + b^{-2})} \sqrt{\frac{\epsilon}{\mu}} \, \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \, e^{i\,k\,z} \, e^{-i\,\omega\,t} .$$

(5)

$$\vec{K} = \frac{c}{4\pi} \, \hat{n} \times \vec{H}$$

Let us choose the convention that $\hat{n}$ points out of the conductor (into the wave guide). Then,

$$\text{For the } (x-z) = \text{plane } y = 0, b : \quad \hat{n} = \pm \hat{y} , \quad \frac{4\pi}{c} \vec{K} = \mp \hat{z} \, H^x ,$$

$$\text{For the } (y-z) = \text{plane } x = 0, a : \quad \hat{n} = \pm \hat{x} , \quad \frac{4\pi}{c} \vec{K} = \pm \hat{z} \, H^y .$$

32. Cubic cavity oscillator (E.99).

$$\vec{E} \;=\; \hat{x} \, E_0 \, \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{a}\right) \, e^{-i\omega t} .$$

(1)

$$\vec{H} = -\frac{i\,c}{\omega\,\mu}\,\nabla\times\vec{E} \; = $$
$$-\,\frac{i\,c}{\omega\,\mu}\,E_0\,\frac{\pi}{a}\,\left[-\hat{z}\,\cos\left(\frac{\pi y}{a}\right)\,\sin\left(\frac{\pi z}{a}\right) + \hat{y}\,\sin\left(\frac{\pi y}{a}\right)\,\cos\left(\frac{\pi z}{a}\right)\right]\,e^{-i\omega t}\,.$$

(2)

$$\nabla\cdot\vec{E} = 0 \quad \text{as } \vec{E} \text{ does not depend on } x \text{ and } \vec{E}\sim\hat{x}\,,$$

$$\nabla\cdot\vec{H}\sim\left[-\cos\left(\frac{\pi y}{a}\right)\,\cos\left(\frac{\pi z}{a}\right) + \cos\left(\frac{\pi y}{a}\right)\,\cos\left(\frac{\pi z}{a}\right)\right]\,e^{-i\omega t} = 0\,.$$

$$\nabla\times\vec{H} = -\frac{i\,c}{\omega\,\mu}\,E_0\,\frac{\pi^2}{a^2}\,\left[\hat{x}\,\sin\left(\frac{\pi y}{a}\right)\,\sin\left(\frac{\pi z}{a}\right) + \hat{x}\,\sin\left(\frac{\pi y}{a}\right)\,\sin\left(\frac{\pi z}{a}\right)\right]\,e^{-i\omega t}$$
$$= -\frac{i\,c}{\omega\,\mu}\,\frac{2\,\pi^2}{a^2}\,\vec{E} \;=\; -\frac{i\,\omega}{c}\,\epsilon\,\vec{E}$$

where the result of (3) below was used in the last step. BCs: From $\sin(\pi a/a) = \sin(\pi) = 0$ we see that $\vec{E}|_S = 0$ holds on the boundaries parallel to $\vec{E}$ and on the boundaries where $\vec{H}$ has a perpendicular component $\vec{H}_\perp|_S = 0$ holds.

(3)

$$0 = \left(\nabla^2 - \frac{\mu\epsilon}{c^2}\,\frac{\partial^2}{\partial t^2}\right)\,\vec{E} = \left(-\frac{\pi^2}{a^2} - \frac{\pi^2}{a^2} + \frac{\mu\epsilon}{c^2}\,\omega^2\right)\,\vec{E} \;\Rightarrow\; \omega = \sqrt{\frac{2}{\mu\epsilon}}\,\frac{c\,\pi}{a}\,.$$

(4) Surface currents:

$$\vec{K} = \frac{c}{4\pi}\,\hat{n}\times\vec{H}\,.$$

Top, bottom $(x{-}y){-}$plane, $z = 0, a$:

$$\vec{K} = \pm\frac{c}{4\pi}\,\hat{z}\times\vec{H} = \pm\hat{x}\,\frac{i\,c^2}{4\omega\mu\epsilon a}\,\sin\left(\frac{\pi y}{a}\right)\,e^{-i\omega t}\,,$$

Front, back $(z{-}x){-}$plane, $y = 0, a$:

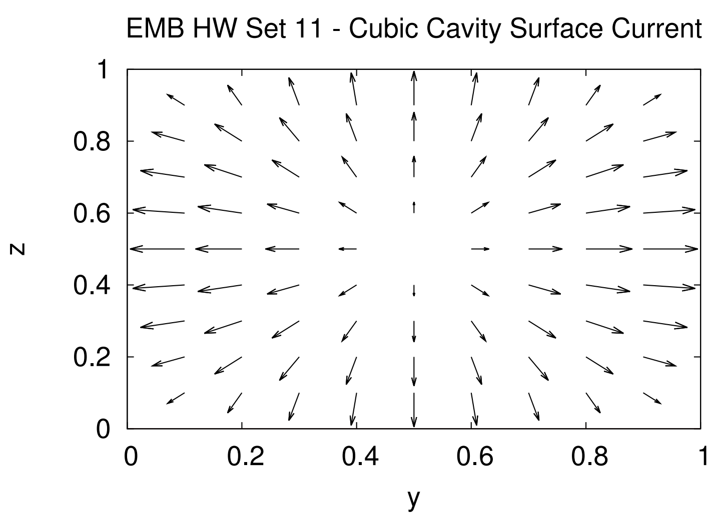
$$\vec{K} = \pm\frac{c}{4\pi}\,\hat{y}\times\vec{H} = \mp\hat{x}\,\frac{i\,c^2}{4\omega\mu\epsilon a}\,\sin\left(\frac{\pi y}{a}\right)\,e^{-i\omega t}\,,$$

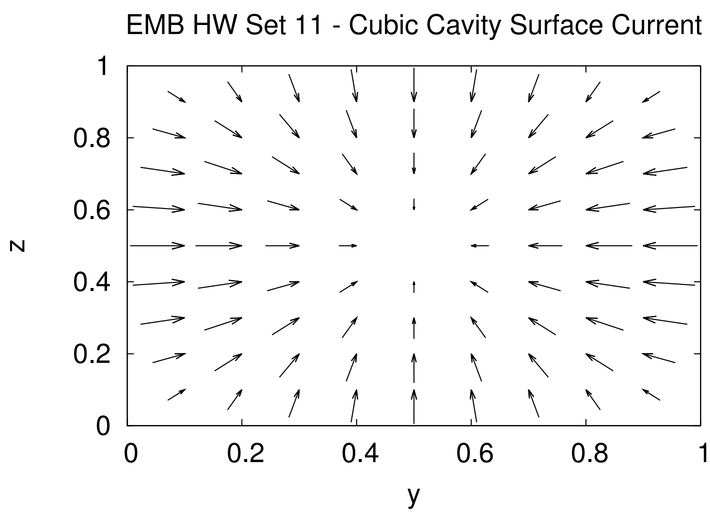
Left, right $(y-z)-$plane, $x = 0, a$:

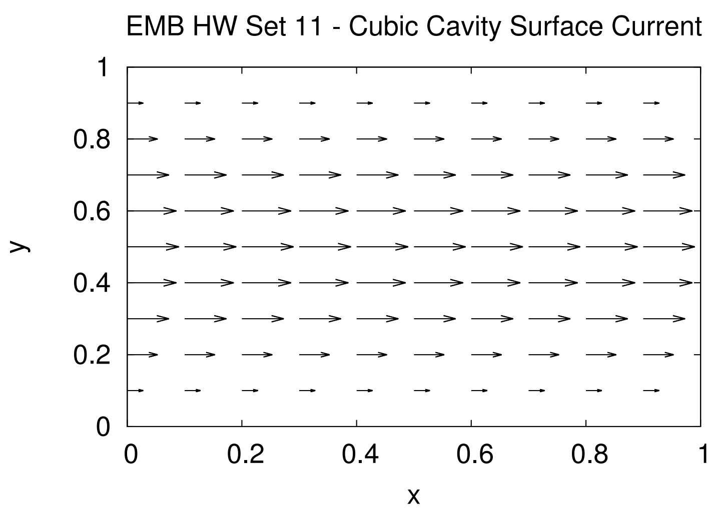
$$\vec{K} = \pm \frac{c}{4\pi}\, \hat{x} \times \vec{H} =$$
$$\mp \frac{i\, c^2}{4\omega\mu\epsilon a} \left[\hat{y} \cos\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{a}\right) + \hat{z} \sin\left(\frac{\pi y}{a}\right) \cos\left(\frac{\pi z}{a}\right) \right] e^{-i\omega t}\, .$$

Surface charge density in the $(y-z)-$plane :

$$\sigma = \frac{1}{4\pi} \frac{2\pi}{\omega^2 \mu} E_0 \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{\pi z}{a}\right) e^{-i\omega t}\ .$$




EMB HW Set 11 - Cubic Cavity Surface Current
z
y
0
0.2
0.4
0.6
0.8
1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