Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 10

27. Principal value integral (E.93).

A.

(4) Calculation of the principal value integral using the first contour becomes:

$$f(\omega_R) = \frac{1}{2\pi\, i} \lim_{\eta\to 0} \left[\int_{-\infty+i\eta}^{\omega_R-\eta+i\eta} \frac{f(z)\,dz}{z-\omega_R+i\eta} + \int_{\omega_R+\eta+i\eta}^{+\infty+i\eta} \frac{f(z)\,dz}{z-\omega_R+i\eta} \right.$$
$$\left. + \int_{C_+(\omega_R,\eta)} \frac{f(z)\,dz}{z-\omega_R+i\eta} \right] = \frac{1}{2\pi i}\, P \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-\omega_R} + \frac{1}{2}\, f(\omega_R)$$
$$\Rightarrow \frac{1}{2\pi i}\, P \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-\omega_R} = \frac{1}{2}\, f(\omega_R)\,.$$

(5) Calculation of the principal value integral using the second contour:

$$0 = \frac{1}{2\pi\, i} \lim_{\eta\to 0} \left[\int_{-\infty+i\eta}^{\omega_R-\eta+i\eta} \frac{f(z)\,dz}{z-\omega_R+i\eta} + \int_{\omega_R+\eta+i\eta}^{+\infty+i\eta} \frac{f(z)\,dz}{z-\omega_R+i\eta} \right.$$
$$\left. + \int_{C_-(\omega_R,\eta)} \frac{f(z)\,dz}{z-\omega_R+i\eta} \right] = \frac{1}{2\pi i}\, P \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-\omega_R} - \frac{1}{2}\, f(\omega_R)$$
$$\Rightarrow \frac{1}{2\pi i}\, P \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-\omega_R} = \frac{1}{2}\, f(\omega_R)\,.$$

As it has to be, the results for the principal value integral agree.

B. We assume that $g(\tau)$ is for small $\tau > 0$ defined by the Taylor expansion

$$g(\tau) = \sum_{n=1}^{\infty} \frac{\tau^n}{n!}\, g^{(n)}(0) \tag{0.1}$$

to find the asymptotic expansion of $\epsilon_0(\omega)$ for large $|\omega|$

$$\epsilon_0(\omega) - 1 = \sum_{n=1}^{\infty} \int_0^{\infty} d\tau\, \frac{\tau^n}{n!}\, g^{(n)}(0)\, e^{i\,\omega\,\tau} = \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} \left(\frac{d}{i\, d\omega}\right)^n \int_0^{\infty} d\tau\, e^{i\,\omega\,\tau},$$

where all integrals are for ω in the upper complex plane. Hence, we get

$$\epsilon_0(\omega) - 1 = \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} \left(\frac{d}{i\,d\omega}\right)^n \left(\frac{-1}{i\,\omega}\right) = \sum_{n=1}^{\infty} g^{(n)}(0) \left(\frac{-1}{i\,\omega}\right)^{n+1}.$$

So, we have the fall-off behavior $\mathcal{O}(|\omega|^{-2})$ for $|\omega| \to \infty$. In particular, in the limit $\omega_I \to 0^+$ we get

$$\mathrm{Re}\,[\epsilon_0(\omega_R) - 1] = \mathcal{O}(\omega_R^{-2}) \quad \text{and} \quad \mathrm{Im}\,[\epsilon_0(\omega_R) - 1] = \mathcal{O}(\omega_R^{-3})\,.$$

28. A relation for the group velocity (E.92).

Using the index of refraction, the dispersion relation reads

$$\omega(k) \;=\; \frac{c\,k}{n(k)}\,.$$

Now, for the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{c\,n}{n^2} - \frac{c\,k}{n^2}\,\frac{dn}{d\omega}\,\frac{d\omega}{dk} \Rightarrow n^2\,v_g + c\,k\,n'\,v_g = c\,n \quad \text{with} \quad n' = \frac{dn}{d\omega}\,.$$

$$v_g \;=\; \frac{c\,n}{n^2 + c\,k\,n'}\,.$$

29. Kramers-Kronig relation in a model (E.94).

(1) The poles of $\epsilon(\omega)$ are at

$$\omega_{1/2} = -\frac{i\gamma_j}{2} \pm \sqrt{-\left(\frac{\gamma_j}{2}\right)^2 + \omega_j^2}\,.$$

As $\omega_j^2 > 0$ we have for the imaginary parts the inequality

$$\mathrm{Im}\,\omega_{1/2} < -\frac{\gamma_j}{2} + \frac{\gamma_j}{2} = 0\,.$$

The poles are in the lower half-plane and the function is analytical in the upper half-plane.

(2) Let $x = \omega^2$ (real ω). We have to differentiate the expression

$$\frac{f}{g} = \frac{\omega_j^2 - x}{(x - \omega_j^2)^2 + \gamma_j^2\,x}$$

with respect to x. We use the formula

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

the denominator g^2 is positive (already g is positive) and the numerator is $-[(x-\omega_j^2)^2+\gamma_j^2\,x]-[2(x-\omega_j^2)+\gamma_j^2]\,(\omega_j^2-x)=x^2-2\omega_j^2\,x-\omega_j^2\gamma_j^2+\omega_j^4\,.$ For the maximum the numerator has to be zero and the solutions

$$x_0=\omega_0^2=\omega_j^2\pm\gamma_j\,\omega_j$$

are found. For $\gamma_j<\omega_j$ and $0\le\omega<\omega_{j,\max}=\sqrt{\omega_j^2-\gamma_j\,\omega_j}$ the numerator $f'g-g'f=(\omega^2-\omega_j)^2-\gamma_j^2\omega_j^2$ as well as the denominator g^2 are positive. So, f/g is increasing and we have a maximum. For $\omega_{j,\max}<\omega<\sqrt{\omega_j^2+\gamma_j\omega_j}$ the ratio f/g is then decreasing with a minimum at the second solution of $f'g-g'f=0$. In between the imaginary part takes its maximum at $\omega=\omega_j$.