Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 4:

8. Generalized Residue Theorem.

(a) See appendix D.

(b) As $f(z)$ is analytical in S, we can transform the integration contour C into a small circle about z_0 without changing the value of the integral. Using the Taylor expansion we have

$$\frac{1}{2\pi i}\oint_C \frac{dz\, f(z)}{(z-z_0)^n} = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_C \frac{dz\,(z-z_0)^k\, f^k(z_0)}{k!\,(z-z_0)^n}$$

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{1}{k!}\oint_C \frac{dz\, f^k(z_0)}{(z-z_0)^{n-k}} \quad \text{with} \quad f^k(z_0) = \left.\frac{d^k f(z)}{dz^k}\right|_{z=z_0}.$$

Using the previously derived result

$$\oint_C \frac{dz}{z^n} = \begin{cases} 2\pi i & \text{for} \quad n=1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

we find

$$\oint_C \frac{dz\, f^k(z_0)}{(z-z_0)^{n-k}} = \begin{cases} 2\pi i\, f^k(z_0) & \text{for} \quad n-k=1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Therefore, only $k=n-1$ contributes to the sum for $n = 1, 2, 3, \dots$ and gives

$$\frac{1}{2\pi i}\oint_C \frac{dz\, f(z)}{(z-z_0)^n} = \frac{f^{n-1}(z_0)}{(n-1)!}\,.$$

9. Green function by Fourier transformation.

We use the notation $\vec{\xi} = \vec{x} - \vec{y}$ and introduce the Fourier transformation by

$$G(\vec{x}-\vec{y}\,) = G(\vec{\xi}\,) = \frac{1}{(2\pi)^3}\int d^3p\, e^{-i\,\vec{p}\cdot\vec{\xi}}\, \widetilde{G}(\vec{p}\,)$$

and

$$\delta(\vec{\xi}\,) = \nabla_x^2 G(\vec{\xi}\,) = \frac{1}{(2\pi)^3}\int d^3p\,(-\vec{p}\,)^2\, e^{-i\,\vec{p}\cdot\vec{\xi}}\, \widetilde{G}(\vec{p}\,) \;\Rightarrow\; \widetilde{G}(\vec{p}\,) = -\frac{1}{\vec{p}^{\,2}}\,.$$

We use spherical coordinates with $d^3p = p^2\,d\phi\,dx$, $x = \cos(\theta)$, where θ is the angle between $\vec{p}$ and $\vec{\xi}$, to evaluate the integral

$$G(\vec{\xi}\,) = -\frac{1}{(2\pi)^3}\int d^3p\,\frac{e^{-i\,\vec{p}\cdot\vec{\xi}}}{\vec{p}^{\;2}} = -\frac{1}{(2\pi)^2}\int_0^\infty dp\int_{-1}^{+1} dx\,e^{-i\,(p\,\xi)\,x}\,,\ \ \xi = |\vec{\xi}\,|\,,$$

where we performed the $d\phi$ integration. Now the dx integration is also elementary

$$G(\vec{\xi}\,) = \frac{i}{(2\pi)^2\xi}\int_0^\infty \frac{dp}{p}\left[e^{-i\,(p\,\xi)\,x}\right]_{-1}^{+1} = -\frac{1}{2\pi^2\xi}\int_0^\infty dp\,\frac{\sin(p\xi)}{p}$$

Using

$$\int_0^\infty dp\,\frac{\sin(p\xi)}{p} \;=\; \frac{\pi}{2}$$

the result is (as expected)

$$G(\vec{\xi}\,) = -\frac{1}{4\pi\,\xi} = -\frac{1}{4\pi\,|\vec{x}-\vec{y}\,|}\,.$$

It remains to calculate the integral

$$I \;=\; \int_0^\infty dx\,\frac{\sin(a\,x)}{x} \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} dx\,\frac{\sin(x)}{x}$$

where the last equality follows from the substitution $x' = ax$ and the fact that the integrand is symmetric under $x \to -x$. Although the integrand is regular at $x = 0$, $\lim_{x\to 0}[\sin(x)/x] = 1$, the integral can be evaluated using the residue theorem by writing

$$\frac{\sin(x)}{x} \;=\; \frac{1}{2\,i}\lim_{\epsilon\to 0^+}\left[\frac{e^{+ix}}{x-i\,\epsilon} - \frac{e^{-ix}}{x-i\,\epsilon}\right]\,,$$

where the notation $\epsilon \to 0^+$ stands for $\epsilon > 0$, $\epsilon to 0$. The integral becomes

$$I \;=\; \frac{1}{4\,i}\lim_{\epsilon\to 0^+}\left[\int_{-\infty}^{+\infty} dx\,\frac{e^{+ix}}{x-i\,\epsilon} - \int_{-\infty}^{+\infty} dx\,\frac{e^{-ix}}{x-i\,\epsilon}\right] \;=\; \lim_{\epsilon\to 0^+}\,\left[I_1(\epsilon) + I_2(\epsilon)\right)\,.$$

We can evaluate $I_1(\epsilon)$ by closing on a staple in the upper and $I_2(\epsilon)$ on a staple in the lower complex planes, so that the staples to not contribute.

E.g.,

$$
\begin{aligned}
I_1(\epsilon) \;=\; \frac{1}{4\,i}\lim_{L\to\infty} & \left[\int_{-L}^{+L} dx\,\frac{e^{+ix}}{x-i\,\epsilon} + \int_0^{+L} i\,dy\,\frac{e^{-y}\,e^{+iL}}{L-i\,\epsilon}\right. \\
+ & \left.\int_{+L}^{-L} dx\,\frac{e^{-L}\,e^{+ix}}{x-i\,\epsilon} + \int_{+L}^{0} i\,dy\,\frac{e^{-y}\,e^{-iL}}{-L-i\,\epsilon}\right]
\end{aligned}
$$

where the staple integral disappear for $L \to \infty$. Note that the sign in $\exp(ix)$ determines whether we have to close in the upper or lower plane. Now we are free to choose the ϵ limit to be from $\epsilon > 0$. The residue theorem gives then

$$\lim_{\epsilon\to 0^+} I_1(\epsilon) = \frac{2\pi\,i}{4\,i} = \frac{\pi}{2} \quad \text{and} \quad I_2(\epsilon) = 0\,.$$

The latter equality holds because there is no pole in the lower plane (with the choice $\epsilon < 0$ it is the other way round, leading to the same result). This proves $I = \pi/2$.

10. Current density of a point charge.

Let x^0 be given. Because $r^0(\tau)$ is a monotonically increasing function, there will be a unique solution

$$x^0 - r^0(\tau^0) = 0 \quad \text{with} \quad U^0(\tau^0) = \left.\frac{dr^0}{d\tau}\right|_{\tau=\tau^0} \neq 0\ .$$

Then

$$\delta\left[x^0 - r^0(\tau)\right] = \frac{1}{|U^0(\tau^0)|}\,\delta(\tau^0-\tau)$$

and the components of

$$j^\alpha(x) \;=\; q\int c\,d\tau\,U^\alpha(\tau)\,\delta^{(4)}\left[x - r(\tau)\right]$$

become (note $U^0(\tau^0) = \gamma(\tau^0)\,c > 0$)

$$j^\alpha(x) \;=\; q\,c\,\frac{U^\alpha(\tau^0)}{U^0(\tau^0)}\,\delta^{(3)}\left[\vec{x} - \vec{r}(\tau^0)\right]$$

where $\tau^0 = \tau^0(x^0)$. As $U^\alpha(\tau^0) = \gamma(\tau^0)\,u^\alpha(\tau^0)$, $u^0 = c$. The final result is

$$j^0(x) = q\,c\,\delta^{(3)}\left[\vec{x} - \vec{r}(\tau^0)\right]\,, \qquad \vec{j}(x) = q\,\vec{u}\,\delta^{(3)}\left[\vec{x} - \vec{r}(\tau^0)\right]\ .$$