Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 2:

2. Magnetic dipoles.

The magnetic dipole approximation holds

$$\vec{B}(\vec{r}) = \frac{3\hat{r}\,(\hat{r}\cdot\vec{m}\,) - \vec{m}}{|\,\vec{r}\,|^3}\,,\quad \hat{r} = \frac{\vec{r}}{r}\,,\quad r = |\,\vec{r}\,|\;.$$

(1) Now we consider $\vec{m}_1$ in the field of $\vec{m}_2$, $\vec{r} = \vec{r}_1 - \vec{r}_2$:

$$U = -\vec{m}_1\cdot\vec{B}_2 = -\left(\frac{3\,(\vec{m}_1\cdot\hat{r})\,(\,\hat{r}\cdot\vec{m}_2\,) - \vec{m}_1\cdot\vec{m}_2}{|\,\vec{r}\,|^3}\right)\,,$$

(2) The force acting on either of the magnets follows from

$$\vec{F} = -\nabla U\,,\qquad \nabla|\vec{r}| = \hat{r} = \frac{\vec{r}}{r}\;.$$

We keep the position of $\vec{m}_2$ fixed and calculate the force acting on $\vec{m}_1$ at the position $\vec{r}$:

$$\begin{aligned}\vec{F} &= +\nabla\frac{3\,(\vec{m}_1\cdot\vec{r})\,(\,\vec{r}\cdot\vec{m}_2\,)}{|\,\vec{r}\,|^5} - \nabla\frac{\vec{m}_1\cdot\vec{m}_2}{|\,\vec{r}\,|^3}\\ &= \frac{3\,\vec{m}_1\,(\,\vec{r}\cdot\vec{m}_2\,)}{|\,\vec{r}\,|^5} + \frac{3\,\vec{m}_2\,(\,\vec{r}\cdot\vec{m}_1\,)}{|\,\vec{r}\,|^5} - 15\,\vec{r}\,\frac{(\vec{m}_1\cdot\vec{r})\,(\,\vec{r}\cdot\vec{m}_2\,)}{|\,\vec{r}\,|^7} + 3\,\vec{r}\,\frac{\vec{m}_1\cdot\vec{m}_2}{|\,\vec{r}\,|^5}\;.\end{aligned}$$

(3) The rest position is at $U = U_{\rm min}$. Let us choose coordinates, so that $\hat{r} = \hat{z}$. Then

$$U = \frac{-2\,m_1^z\,m_2^z + m_1^x\,m_2^x + m_1^y\,m_2^y}{|\,\vec{r}\,|^3}\;.$$

We find negative contributions for $\mathrm{sign}(m_1^z) = \mathrm{sign}(m_2^z)$, $\mathrm{sign}(m_1^x) = -\mathrm{sign}(m_2^x)$ and $\mathrm{sign}(m_1^y) = -\mathrm{sign}(m_2^y)$, which are therefore assumed to be chosen by the system. In particular, this leads to

$$\vec{m}_2^{\perp} = -\alpha\,\vec{m}_1^{\perp}\,,\qquad \alpha > 0$$

for the components perpendicular to $\hat{z}$ (i.e. they are vectors in the x–y plane). For $i = 1,2$ let $m_i^{\perp} = |\vec{m}_i^{\perp}|$. Then (all terms give negative contributions)

$$U = \frac{-2\,m_1^z\,m_2^z - m_1^{\perp}, m_2^{\perp}}{|\,\vec{r}\,|^3}\;.$$

With $m_i = |\vec{m}_i|$ for $i = 1, 2$ we can write

$m_1^z = m_1 \cos(\alpha)\,,\ m_1^\perp = m_1 \sin(\alpha)$ and $m_2^z = m_2 \cos(\beta)\,,\ m_2^\perp = m_2 \sin(\beta)$

and obtain

$$U \;=\; \frac{-2\, m_1\, m_2\, \cos(\alpha)\, \cos(\beta) - m_1\, m_2\, \sin(\alpha)\, \sin(\beta)}{|\,\vec{r}\,|^3}\;.$$

So, we have to minimize $-2\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Using well-known trigonometric equations, this becomes

$$-\frac{3}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)\,.$$

As the maximum of $\cos(\phi)$ is at $\phi = 0$, the only solution for the minimum of U is $\alpha = \beta = 0$. Hence, $\vec{m}_1$ is parallel to $\vec{m}_2$ along the $\hat{r}$ axis.
In this configuration there is an attractive force of magnitude

$$F_{\min} = |3 + 3 - 15 + 3|\,\frac{m_1\, m_2}{r^4} = 6\,\frac{m_1\, m_2}{r^4}$$

between the dipoles. The magnetic field of this configuration is

$$\begin{aligned}\vec{B}(\vec{r}\,) &= \sum_{i=1}^{2} \frac{3\,(\vec{r} - z_i\,\hat{z})\, m_i\,\hat{z}\cdot(\vec{r} - z_i\,\hat{z}) - m_i\,\hat{z}\,(\vec{r} - z_i\,\hat{z})^2}{|\vec{r} - z_i\,\hat{z}|^5} \\ &= \sum_{i=1}^{2} \frac{3\,(\vec{r} - z_i\,\hat{z})\, m_i\,(r\,\cos\theta - z_i) - m_i\,\hat{z}\,[r^2 - 2\,r\,z_i\,\cos\theta + (z_i)^2]}{[r^2 - 2\,r\,z_i\,\cos\theta + (z_i)^2]^{5/2}}\,.\end{aligned}$$

3. Magnetic moment of a disk.

(1) The charge density on the perimeter is $\lambda = q/(2\pi\,R)$, so the current density is (with $r = |\vec{r}|$)

$$\vec{J}(\vec{r}) = \omega\,R\,\lambda\,\delta(\cos\theta)\,R^{-1}\delta(r-R)\,\hat{\phi}\,,$$

and the current becomes

$$\vec{I} = \int_{-1}^{+1} d\cos\theta \int_0^\infty r\,dr\,\vec{J}(\vec{r}) = \omega\,R\,\lambda\,\hat{\phi}\,.$$

The magnetic moment is now

$$\vec{\mu} = \frac{1}{2c} \int \vec{r} \times \vec{J}\, d^3x$$
$$= \frac{2\pi}{2c}\, \omega\, \lambda\, (-\hat{\theta}) \int_{-1}^{+1} d\cos\theta\, \delta(\cos\theta) \int_0^\infty r^3 dr\, \delta(r - R)$$
$$= \frac{\pi}{c}\, \omega\, R^3\, \lambda\, \hat{z} \;=\; \frac{1}{2c}\, \omega\, R^2\, q\, \hat{z}\, .$$

(2) Let σ_m be the mass density of the disk. The angular momentum is

$$\vec{L} = L_z\, , \quad L_z = 2\pi\, \sigma_m\, \omega \int_0^R r^3 dr = \frac{\pi}{2}\, \sigma_m\, \omega\, R^4\, ,$$

and the total mass is given by

$$M = 2\pi \int_0^r r dr\, \sigma_m = \pi\, R^2\, \sigma_m \quad \Rightarrow L_z = \frac{M}{2}\, \omega\, R^2\, .$$

Therefore, the magnetic moment can be written

$$\mu_z = g\, \frac{q}{2M\, c}\, L_z \quad \text{with} \quad g = 2$$

which is the g factor of the electron.