Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 12

33. Cylindrical waveguide, TE_{11} mode power transmission.

The cutoff frequency follows as the $d \to \infty$ limit from our discussion of the cylindrical cavity,

$$\omega_{11} = \frac{1.841\, c}{\sqrt{\mu\epsilon}\, R} = \frac{\gamma'_{11}}{R}\ .$$

Using the time-averaged Poynting vector

$$\vec{S} \ = \ \frac{c}{8\pi} \mathrm{Re} \left(\vec{E} \times \overline{\vec{H}} \right)$$

and the TE mode

$$H^z = \psi\, e^{ikz}\,, \quad \left.\frac{\partial \psi}{\partial n}\right|_S = 0\,, \quad \vec{H}_t = \frac{ik}{\gamma^2}\, \nabla_t \psi\,, \quad \vec{E}_t = -Z\, hatz \times \vec{H}_t\,,$$

where $Z = \mu\omega/(ck)$ is the wwave impedance for the TE mode, one finds

$$\vec{S} \ = \ \frac{\omega\, k\, \mu}{8\pi\, \gamma^4} \left[\hat{z}\, |\nabla_t \psi|^2 - i\, \frac{\gamma^2}{k}\, \overline{\psi}\, \nabla_t \psi \right] .$$

We are only interested in the longitudinal energy $\sim \hat{z}$ and the power flow is obtained by integrating over the cross-sectional area:

$$P \ = \ \int_A \hat{z} \cdot \vec{S} \ = \ \frac{\omega\, k\, \mu}{8\pi\, \gamma^4} \int_A (\overline{\nabla_t \psi}) \cdot (\nabla_t \psi)\, da\,.$$

By means of Green's firt identity applied to two dimensions, P becomes

$$P \ = \ \frac{\omega\, k\, \mu}{8\pi\, \gamma^4} \left[\oint_C \overline{\psi}\, \frac{\partial \psi}{\partial n}\, dl - \int_A \overline{\psi}\, \nabla_t^2 \psi\, da \right] ,$$

where the first integral is zero due to the BC. Using the wave equation for the second integral, the transmitted power for mode γ_λ^2 is

$$P \ = \ \frac{\omega\, k_\lambda\, \mu}{8\pi\, \gamma_\lambda^4} \int_A \overline{\psi}\, \gamma_\lambda^2\, \psi\, da \ = \ \frac{c\, \mu}{8\pi\, \sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda} \right)^2 \left(1 - \frac{\omega_\lambda^2}{\omega^2} \right)^{1/2} \int_A \overline{\psi}\, \psi\, da\,,$$

where in the last step γ_λ^2 has been eliminated in favor of the cutoff frequency, $\gamma_\lambda = \sqrt{\mu\epsilon}\, \omega_\lambda / c$ and $k_\lambda = \sqrt{\mu\epsilon} \sqrt{\omega^2 - \omega_\lambda^2}/c$. For the TE_{11} mode

$$P = \frac{c\, \mu}{8\pi\, \sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_{11}} \right)^2 \left(1 - \frac{\omega_{11}^2}{\omega^2} \right)^{1/2} \int_A da\, \overline{\psi}\, \psi$$

and

$$\psi = H_0\, J_1(\gamma'_{11}\,\rho)\, e^{i\,\phi}$$

for the mode in question. Thus

$$P = \frac{c\,\mu}{8\pi\,\sqrt{\mu\epsilon}}\,\left(\frac{\omega}{\omega_{11}}\right)^2\,\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{1/2}\,2\pi\,|H_0|^2\int_0^R \rho\,d\rho\,\left[J_1(\gamma'_{11}\,\rho)\right]^2 \ .$$

Using the integral as given in the problem the final result is

$$P = \frac{c\,\mu}{8\,\sqrt{\mu\epsilon}}\,\left(\frac{\omega}{\omega_{11}}\right)^2\,\left(1-\frac{\omega_{11}^2}{\omega^2}\right)^{1/2}\,|H_0|^2\,R^2\,\left(1-\frac{1}{x'^2_{11}}\right)\,\left[J_1(\gamma'_{11}\,R)\right]^2 \ .$$

34. Hertz vector.

(1) We have

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha\,, \qquad (A^\alpha) = \begin{pmatrix}\Phi\\ \vec{A}\end{pmatrix} \ .$$

In the Lorentz gauge $\partial_\alpha A^\alpha = 0$ holds and, therefore,

$$\partial_\alpha F^{\alpha\beta} = \Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = \Box A^\beta = \frac{4\pi}{c}\,J^\alpha \ .$$

In components this reads

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\,\rho\,, \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\,\vec{J}\,.$$

(2) Continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\,(\nabla\cdot\vec{p}) + \nabla\cdot\left(\frac{\partial\vec{p}}{\partial t}\right) = 0 \ .$$

(3) Let $\Phi = a\left(\nabla\cdot\vec{\Pi}\right)$ and $\vec{A} = b\,\partial\vec{\Pi}/\partial t$. Then,

$$a\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left(\nabla\cdot\vec{\Pi}\right) \;=\; -a\,4\pi\,\nabla\cdot\vec{p} \;=\; 4\pi\,\rho \;\Rightarrow\; a \;=\; -1\,,$$

$$b\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial\vec{\Pi}}{\partial t}\right) \;=\; -b\,4\pi\,\frac{\partial\vec{p}}{\partial t} \;=\; -\frac{4\pi}{c}\,\vec{J} \;\Rightarrow\; b \;=\; \frac{1}{c}\,,$$

$$\Phi = -\left(\nabla\cdot\vec{\Pi}\right)\,, \qquad \vec{A} = \frac{1}{c}\,\frac{\partial\vec{\Pi}}{\partial t} \ .$$

The Lorentz condition holds:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \cdot \vec{\Pi}\right) + \frac{1}{c}\,\nabla \cdot \frac{\partial \vec{\Pi}}{\partial t} = 0 \ .$$

(4) For the given Hertz vector we find

$$\Phi = -\nabla \cdot \vec{\Pi} = -p_0\, \cos(\theta)\,\frac{\partial}{\partial r}\left(r^{-1}\, e^{-i\,\omega\, t + i\, k\, r}\right) \approx -\frac{i\,k\,p_0}{r}\, \cos(\theta)\, e^{-i\,\omega\, t+i\,k\,r} \,,$$

$$\vec{A} = \frac{1}{c}\,\frac{\partial \vec{\Pi}}{\partial t} = -\frac{i\,\omega\,p_0}{c\,r}\,\hat{z}\,e^{-i\,\omega\,t+i\,k\,r} \,.$$

(5) The electric and magnetic fields are given by

$$\vec{E} = -\frac{1}{c}\,\frac{\partial \vec{A}}{\partial t} - \nabla\Phi \,, \qquad \vec{B} = \nabla \times \vec{A} \ .$$

Therefore, we have in the far field approximation (use ∇ in spherical coordinates)

$$\vec{E} \approx \frac{\omega^2\, p_0}{c^2\, r}\, \left(-\hat{z} + \cos(\theta)\, \hat{r}\right)\, e^{-i\,\omega\,t+i\,k\,r} \ .$$

Using $\hat{z} = \cos(\theta)\,\hat{r} - \sin(\theta)\,\hat{\theta}$ this is

$$\vec{E} \approx \frac{\omega^2\, p_0}{c^2\, r}\, \sin(\theta)\,\hat{\theta}\, e^{-i\,\omega\,t+i\,k\,r} \ .$$

For the magnetic field we have

$$\vec{B} = -\frac{i\,\omega\,p_0}{c}\,\hat{r} \times \hat{z}\,\frac{\partial}{\partial r}\,\frac{e^{-i\,\omega\,t+i\,k\,r}}{r} \approx -\frac{\omega\,k\,p_0}{c\,r}\, \sin(\theta)\,\hat{\phi}\, e^{-i\,\omega\,t+i\,k\,r} \ .$$

The $\vec{E}$ and $\vec{B}$ fields describe electric dipole radiation.