Electrodynamics B (PHY 5347) Winter/Spring 2017 Solutions

Set 7:

17. Lorentz force and energy-momentum tensor (E.77).

Given is

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu}\,F_{\mu\lambda}\,F^{\lambda\beta} + \frac{1}{4}\,g^{\alpha\beta}\,F_{\mu\lambda}\,F^{\mu\lambda}\right],$$

we have

$$\partial_\alpha\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[\partial^\mu\,\left(F_{\mu\lambda}\,F^{\lambda\beta}\right) + \frac{1}{4}\,\partial^\beta\,\left(F_{\mu\lambda}\,F^{\mu\lambda}\right)\right]$$
$$= \frac{1}{4\pi}\left[F^{\lambda\beta}\,\partial^\mu\,F_{\mu\lambda} + F_{\mu\lambda}\,\partial^\mu\,F^{\lambda\beta} + \frac{1}{2}\,F_{\mu\lambda}\,\partial^\beta\,F^{\mu\lambda}\right].$$

The last term is in this form due to $\left(\partial^\beta\,F_{\mu\lambda}\right)\,F^{\mu\lambda} = F_{\mu\lambda}\,\partial^\beta\,F^{\mu\lambda}$. Using the inhomogeneous Maxwell equation, $\partial^\mu\,F_{\mu\lambda} = (4\pi/c)\,J_\lambda$,

$$\partial_\alpha\Theta^{\alpha\beta} = F^{\lambda\beta}\,\frac{1}{c}\,J_\lambda + \frac{1}{4\pi}\,F_{\mu\lambda}\,\left(\partial^\mu\,F^{\lambda\beta} + \frac{1}{2}\,\partial^\beta\,F^{\mu\lambda}\right).$$

Remembering the homogeneous Maxwell equation in the form

$$\partial^\beta\,F^{\mu\lambda} = -\partial^\mu\,F^{\lambda\beta} - \partial^\lambda\,F^{\beta\mu},$$

our expression becomes

$$\partial_\alpha\Theta^{\alpha\beta} = F^{\lambda\beta}\,\frac{1}{c}\,J_\lambda + \frac{1}{8\pi}\,F_{\mu\lambda}\,\left(\partial^\mu\,F^{\lambda\beta} - \partial^\lambda\,F^{\beta\mu}\right) = F^{\lambda\beta}\,\frac{1}{c}\,J_\lambda = -f^\beta$$

as (first re-naming summation indices, then using anti-symmetry)

$$F_{\mu\lambda}\,\partial^\mu\,F^{\lambda\beta} - F_{\mu\lambda}\,\partial^\lambda\,F^{\beta\mu} = F_{\lambda\mu}\,\partial^\lambda\,F^{\mu\beta} - F_{\mu\lambda}\,\partial^\lambda\,F^{\beta\mu} =$$

$$F_{\mu\lambda}\,\partial^\lambda\,F^{\beta\mu} - F_{\mu\lambda}\,\partial^\lambda\,F^{\beta\mu} = 0\,.$$

The same result is obtained starting with

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[F^\alpha_\lambda\,F^{\lambda\beta} + \frac{1}{4}\,g^{\alpha\beta}\,F_{\mu\lambda}\,F^{\mu\lambda}\right],$$

18. Field energy of a sphere (E.78).

(1) For $\vec{B}=0$ the energy-momentum tensor is given by

$$\Theta^{00}=\frac{1}{8\pi}\vec{E}^{\,2}\,,\qquad \Theta^{i0}=\Theta^{0i}=0\,,\qquad \Theta^{ij}=-\frac{1}{4\pi}E^i\,E^j-g^{ij}\,\vec{E}^{\,2}\,.$$

Therefore,

$$P^0=\int d^3x\,\Theta^{00}=\frac{1}{8\pi}\int d^3x\,\vec{E}^{\,2}\quad\text{and}\quad P^i=\int d^3x\,\Theta^{0i}=0\ .$$

Let q be the total charge on the sphere and we take the center of the sphere at $\vec{r}=0$. By Gauss' law the field is zero inside the sphere and

$$\vec{E}=E(r)\,\hat{r}=\frac{q\,\hat{r}}{r^2}\quad\text{outside}\,.$$

So $P^0(R)$ becomes

$$P^0(R)=\frac{4\pi}{8\pi}\int_R^\infty r^2\,dr\,\frac{q^2}{r^4}=-\frac{q^2}{2}\,\frac{1}{r}\bigg|_R^\infty=\frac{q^2}{2R}\ .$$

(2) Now

$$P^0=m_e\ \Rightarrow\ R=\frac{q_e^2}{2\,m_e\,c^2}$$

where $q_e=4.803\times10^{-10}$ [esu], $m_e=0.911\times10^{-27}$ [g], and $c=2.998\times 10^{10}$ [cm/s]. From this we find

$$R=\frac{(4.803)^2\times10^{-20}}{1.822\times10^{-27}\times(2.998)^2\times10^{20}}\,\text{cm}=1.41\times10^{-13}\,\text{cm}=1.41\,\text{fermi}\ .$$

19. Covariant derivation of the wave equation for fields (E.82).

(1)

$$\begin{aligned}
0\ &=\ \epsilon^{\alpha\beta\gamma\delta}\partial_\beta\partial^{\hat\beta}\ {}^*F_{\alpha\hat\beta}=\frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\hat\beta\hat\gamma\hat\delta}\partial_\beta\partial^{\hat\beta}\,F^{\hat\gamma\hat\delta}=\\
&\frac{1}{2}\,\Big(-\delta^\beta_{\ \hat\beta}\delta^\gamma_{\ \hat\gamma}\delta^\delta_{\ \hat\delta}-\delta^\beta_{\ \hat\gamma}\delta^\gamma_{\ \hat\delta}\delta^\delta_{\ \hat\beta}-\delta^\beta_{\ \hat\delta}\delta^\gamma_{\ \hat\beta}\delta^\delta_{\ \hat\gamma}\\
&\qquad+\delta^\beta_{\ \hat\beta}\delta^\gamma_{\ \hat\delta}\delta^\delta_{\ \hat\gamma}+\delta^\beta_{\ \hat\delta}\delta^\gamma_{\ \hat\gamma}\delta^\delta_{\ \hat\beta}+\delta^\beta_{\ \hat\gamma}\delta^\gamma_{\ \hat\beta}\delta^\delta_{\ \hat\delta}\Big)\,\partial_\beta\partial^{\hat\beta}\,F^{\hat\gamma\hat\delta}=\\
&\frac{1}{2}\,\big(-\partial_\beta\partial^\beta\,F^{\gamma\delta}-\partial_\beta\partial^\delta\,F^{\beta\gamma}-\partial_\beta\partial^\gamma\,F^{\delta\beta}\\
&\qquad+\partial_\beta\partial^\beta\,F^{\delta\gamma}+\partial_\beta\partial^\delta\,F^{\gamma\beta}+\partial_\beta\partial^\gamma\,F^{\beta\delta}\big)\ .
\end{aligned}$$

Using $\partial_\beta F^{\beta\alpha} = 0$ and the antisymmetry of the field tensor, this becomes

$$\partial_\beta \partial^\beta \, F^{\delta\gamma} = \Box \, F^{\delta\gamma} = 0 \,,$$

which is the wave equation for the fields.

(2) We find the wave equations

$$\partial_\beta \partial^\beta \, F^{\delta\gamma} = \Box \, F^{\gamma\delta} = \frac{4\pi}{c} \, \left(\partial^\gamma J^\delta - \partial^\delta J^\gamma \right) \, .$$

with antisymmetric sources on the right-hand side.