Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 4

10. Proper time under rotation.

The velocity v of the satellite is related to the angular frequency ω by $v = \omega\, r$. Therefore,

$$c^2\, d\tau^2 = c^2\, dt^2 - \omega^2 r^2\, dt^2 \Rightarrow \frac{d\tau}{dt} = \sqrt{1 - \frac{\omega^2\, r^2}{c^2}}\ .$$

For the gravitational acceleration a_g we have the equation

$$a_g = \frac{G\, m_e}{r_s^2} = \frac{v^2}{r_s}\,,$$

where r_s is introduced for the radius of the stationary orbit. Eliminating v gives

$$\frac{G\, m_e}{r_s^2} = \omega^2\, r_s = r_s\, \left(\frac{2\pi}{T}\right)^2\ \Rightarrow\ r_s^3 = G\, m_e\, \left(\frac{T}{2\pi}\right)^2\ ,$$

i.e., Kepler's 3rd law. For a stationary satellite, as well as for a point on the equator, we have $T = 24 \times 3600\, [s]$ for the period. Crunching the numbers, the result for r_s is

$$r_s = \left[G\, m_e\, \left(\frac{T}{2\pi}\right)^2\right]^{1/3} = 4.22 \times 10^7\, [m]\,,$$

which is about 6.6 times the earth radius.

From the first equation we find

$$\begin{aligned}\frac{d\tau_s}{d\tau_e} &= \frac{d\tau_s}{d\,t}\,\frac{d\,t}{d\tau_e} = \frac{\sqrt{1-\omega^2\, r_s^2/c^2}}{\sqrt{1-\omega^2\, r_e^2/c^2}}\\ &\approx \left(1 - \frac{1}{2}\,\frac{\omega^2\, r_s^2}{c^2}\right)\,\left(1 + \frac{1}{2}\,\frac{\omega^2\, r_e^2}{c^2}\right) \approx 1 - \frac{1}{2}\,\frac{\omega^2\, r_s^2}{c^2} + \frac{1}{2}\,\frac{\omega^2\, r_e^2}{c^2}\,.\end{aligned}$$

The numerical values are

$$\frac{1}{2}\,\frac{\omega^2\, r_s^2}{c^2} = 5.249 \times 10^{-11}\ \ \text{and}\ \ \frac{1}{2}\,\frac{\omega^2\, r_e^2}{c^2} = 1.194 \times 10^{-12}\,.$$

The combined, measurable, effect is therefore

$$1 - \frac{d\tau_s}{d\tau_e} = \frac{1}{2}\,\frac{\omega^2\, r_s^2}{c^2} - \frac{1}{2}\,\frac{\omega^2\, r_e^2}{c^2} = 5.130 \times 10^{-11}\,,$$

i.e., the clock on the satellite ticks slower.

From gravity we obtain

$$1 - \frac{d\tau_s}{d\tau_e} = -\frac{G\, m_e}{c^2}\,\left(\frac{1}{r_s} - \frac{1}{r_e}\right) = -5.911 \times 10^{-10}\,,$$

which is about ten times larger than the SR effect.