Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 5

(11) Twin travel in 2D Minkowski space continued.

(a) $\beta^3_A = \beta^1_B = 4/5$.

(b) From the addition theorem of velocities we find

$$\beta^3_B = \left(\frac{\beta^1_B + \beta^1_B}{1 + (\beta^1_B)^2}\right) = \frac{8/5}{1 + 16/25} = \frac{40}{41}\,.$$

(c) We give two solutions. For simplicity of the notation we denote time and space coordinates in K_3 by t and x without subscripts for K_3.

First, we may use that the proper time is invariant. So, from

$$t^2 - x^2 = 3^2 = 9 \quad \text{and} \quad x = \beta^3_B\, t = 40\, t/41$$

we get

$$t^2 - (40\, t/41)^2 = 9 \quad \Rightarrow \quad t = \pm 41/3$$

of which we have to discard the negative solution. The corresponding x coordinate follows from $x = \beta^3_B\, t = (40/41) \times (41/3) = 40/3$, i.e., the position of B_0 is given by

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 41/3 \\ 40/3 \end{pmatrix}\,.$$

Second, we work out the Lorentz transformation from K_1 to K_3:

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} = \sqrt{\frac{5^2}{5^2 - 4^2}} = \frac{5}{3} \quad \text{and} \quad \beta\,\gamma = -\frac{4}{3}\,.$$

So, we find for the position of of B_0 in K_3

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\,\gamma \\ -\beta\,\gamma & \gamma \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5/3 & 4/3 \\ 4/3 & 5/3 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 41/3 \\ 40/3 \end{pmatrix}\,.$$

(d) At B_0 the elapsed time on the clock of B is 3.

(e) We may find the coordinates of the space-time point where A meets B again in K_3 by Lorentz transformation from K_1. Shorter is using the observation that B performs the travel from B_0 to the final point in its rest frame in 3 time units. So, we only have to add 3 to the previously calculated position of B_0 and find

$$\begin{pmatrix} 3+41/3 \\ 40/3 \end{pmatrix} = \begin{pmatrix} 50/3 \\ 40/3 \end{pmatrix} .$$

(f) See the first figure of this solution.

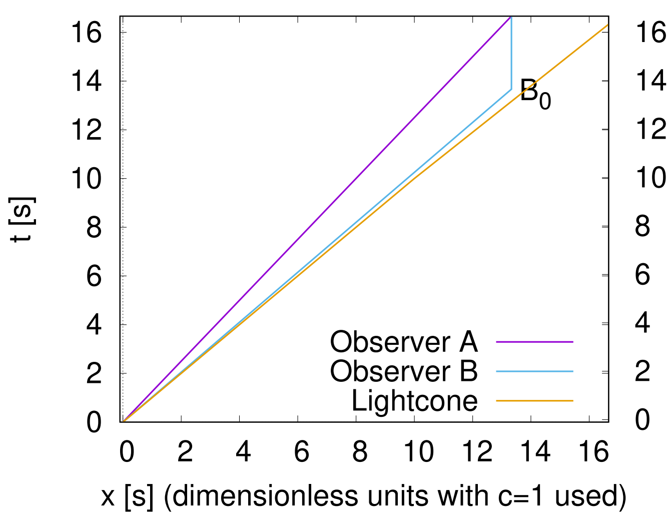

Figure: Travel of the inertial frame K_1 translated to the inertial frame K_3.

(g) The inertial frames K_4 and K_3 differ by the translation $\begin{pmatrix} \triangle t \\ \triangle x \end{pmatrix}$: $\begin{pmatrix} t_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} t_3 \\ x_3 \end{pmatrix} + \begin{pmatrix} \triangle t \\ \triangle x \end{pmatrix}$, where

$$\begin{pmatrix} \triangle t \\ \triangle x \end{pmatrix} = \begin{pmatrix} t_2(B_0) \\ x_2(B_0) \end{pmatrix} - \begin{pmatrix} t_3(B_0) \\ x_3(B_0) \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 41/3 \\ 40/3 \end{pmatrix} = -\begin{pmatrix} 32/3 \\ 40/3 \end{pmatrix} .$$

(h) See the second figure of this solution.

(i) See the third and last figure of this solution.

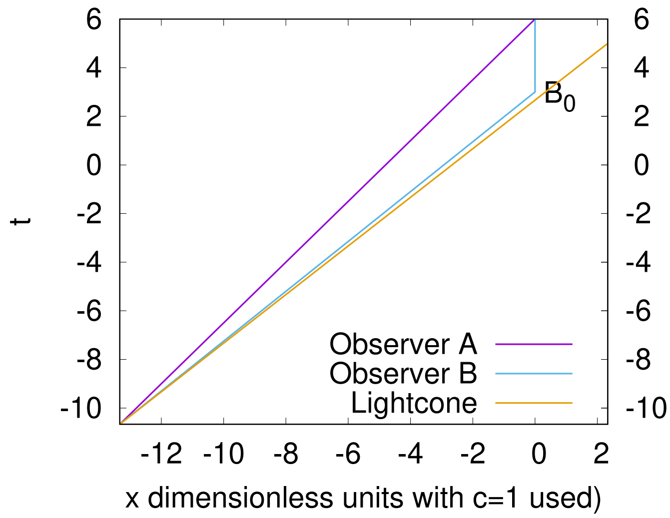

Figure: Travel of the inertial frame K_3 shifted to K_4 by the translation

$$\begin{pmatrix}\triangle t\\ \triangle x\end{pmatrix} = -\begin{pmatrix}32/3\\ 40/3\end{pmatrix}\,, \quad \begin{pmatrix}t_4\\ x_4\end{pmatrix} = \begin{pmatrix}t_3\\ x_3\end{pmatrix} + \begin{pmatrix}\triangle t\\ \triangle x\end{pmatrix}\,.$$

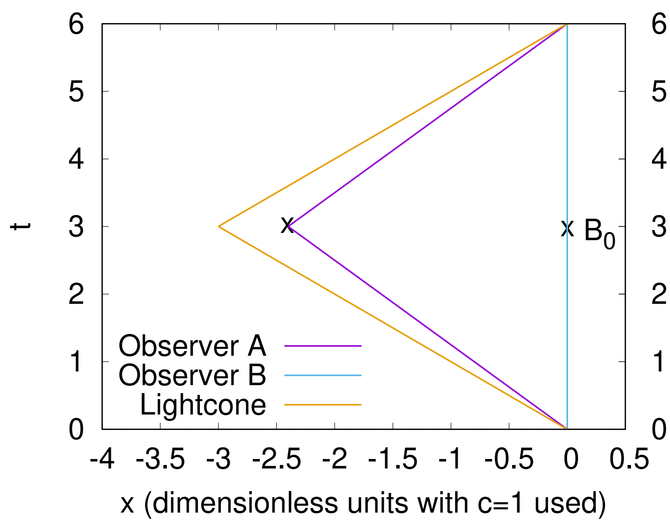

Figure: Inertial frames K_2 and K_4 patched together, so that the travel of B is always in its rest frame. The lines for A meet at $\begin{pmatrix}t\\ x\end{pmatrix} = \begin{pmatrix}3\\ -2.4\end{pmatrix}$.

(j) A travels to the patch point in proper time $3^2 - (2.4)^2 = 3.24$, in K_2 counted from the starting point, and in K_4 counted backward from the terminal point. In K_1 these positions are on the $x = 0$ axis, at $t = 3.24$ for the space-time point from K_2 and at $t = 10 - 3.24 = 6.76$ for the space-time point from K_4.