Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 5

12. Spacetrip Elapsed time on earth.

We only consider the first quarter (five years) of the flight. The other results for the time are the same by symmetry reasons.
Let us denote the inertial frame (IF) "earth" by S_0 and the speed of the spaceship in S_0 by $\beta(\tau)$ (this is equivalent to $\beta(t)$, but more convenient in the problem at hand). We consider a fixed proper time τ_1, $\beta_1 = \beta(\tau_1)$. By S_1 we denote the IF that moves with β_1 away from S and has the spaceship at its origin at proper time τ_1. At τ_1 we have then in S_1

$$(d\beta)_{S_1} = (d\xi)_{S_1} = \frac{g}{c}\, d\tau\,.$$

This translates in S_0 to

$$(d\beta)_{S_0} = \frac{\beta_1 + (d\beta)_{S_1}}{1+\beta_1\,(d\beta)_{S_1}} - \beta_1 = (\beta_1 + (d\beta)_{S_1})\,(1+\beta_1\,(d\beta)_{S_1}) - \beta_1$$

$$= (\beta_1 + (d\beta)_{S_1})\,(1-\beta_1\,(d\beta)_{S_1}) - \beta_1 = \beta_1 + (d\beta)_{S_1} - (\beta_1)^2\,(d\beta)_{S_1} - \beta_1$$

$$= (d\beta)_{S_1} - (\beta_1)^2\,(d\beta)_{S_1} = (d\beta)_{S_1}\,(1-(\beta_1)^2)\,.$$

It is more convenient to work with the rapidity. Because the rapidity is simply additive, we have in S_0

$$(d\xi)_{S_0} = \xi_1 + (d\xi)_{S_1} - \xi_1 = (d\xi)_{S_1} = d\xi\,,$$

where we jsut dropped the label S_1 at the last equal sign. With the initial condition $\zeta(0) = 0$ this integrates to

$$\zeta = \int_0^{\zeta} d\zeta' = \frac{g}{c}\int_0^{\tau} d\tau' = \frac{g}{c}\,\tau\,.$$

The age of the twin on earth follows from $d\tau = dt/\cosh(\zeta)$:

$$\int_0^t dt' = t = \int_0^{\tau}\cosh[\zeta(\tau')]\,d\tau' = \int_0^{\tau}\cosh\left[\frac{g}{c}\,\tau'\right]\,d\tau' = \frac{c}{g}\,\sinh\left[\frac{g}{c}\,\tau\right]\,.$$

Inserting $\tau = 5\,\text{years} = 5\times 365\times 24\times 3600\,[s]$, $c = 3\times 10^8\,[m/s]$ and $g = 9.81\,[m/s^2]$, we find $t = 84.1$ years for a quarter of the trip and 337 years for the entire trip.

The maximum distance traveled from earth is $2\,x$ with

$$x = \int_0^t v(t)\,dt = \int_0^\tau \cosh\left[\zeta(\tau)\right]\,v(\tau)\,d\tau = c\,\int_0^\tau \sinh\left(\frac{g\tau}{c}\right)\,d\tau$$

$$= \frac{c^2}{g}\,\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]\,.$$

Numerical values: $x = 83.4$ light years, maximum distance $= 166.8$ light years.