Special and General Relativity (PHZ 4601/5606) Fall 2018 Solutions

Set 6

17. Electromagnetic field tensor in $\vec{e}$ and $\vec{b}$ fields.

A. From the J^4 component we get

$$E^{44}{}_{,4} + E^{14}{}_{,1} + E^{24}{}_{,2} + E^{34}{}_{,3} = \frac{4\pi}{c}\, J^4 = 4\pi\,\rho = \nabla \cdot \vec{e} = \frac{\partial e^1}{\partial x^1} + \frac{\partial e^2}{\partial x^2} + \frac{\partial e^3}{\partial x^3}\,.$$

Therefore, $E^{14} = e^1$, $E^{24} = e^2$, $E^{34} = e^3$, and $E^{44} = 0$ by anti-symmetry. From the J^1 component we get

$$E^{41}{}_{,4} + E^{11}{}_{,1} + E^{21}{}_{,2} + E^{31}{}_{,3} = \frac{4\pi}{c}\, J^1 = \frac{\partial b^3}{\partial x^2} - \frac{\partial b^2}{\partial x^3} - \frac{\partial e^1}{\partial x^4}\,.$$

Therefore, $E^{41} = -e^1$ consistent with $E^{14} = e^1$, $E^{11} = 0$ by anti-symmetry, $E^{21} = b^3$ and $E^{31} = -b^2$. From the J^2 component we get

$$E^{42}{}_{,2} + E^{12}{}_{,1} + E^{22}{}_{,2} + E^{32}{}_{,3} = \frac{4\pi}{c}\, J^2 = \frac{\partial b^1}{\partial x^3} - \frac{\partial b^3}{\partial x^3} - \frac{\partial e^2}{\partial x^4}\,.$$

Therefore, $E^{42} = -e^2$, $E^{12} = -b^3$ both consistent with anti-symmetry, $E^{22} = 0$ by anti-symmetry and (new) $E^{32} = b^1$.

Using anti-symmetry all components are now determined and the $E^{\alpha\beta}$ field tensor reads

$$\left(E^{\alpha\beta}\right) = \begin{pmatrix} 0 & -b^3 & b^2 & e^1 \\ b^3 & 0 & -b^1 & e^2 \\ -b^2 & b^1 & 0 & e^3 \\ -e^1 & -e^2 & -e^3 & 0 \end{pmatrix}\,.$$

The remaining equations from $\beta = 3$ are not needed, but can be used for consistency checks.