Special and General Relativity (PHZ 4601/5606) Fall 2018 Solutions Set 10

23. Partial Derivatives.

(a) From $x' = x^2\, y^2$ and $y' = x^2/y^2$ we get

$$x^4 = x'\, y' \quad \text{and} \quad y^4 = \frac{x'}{y'} \quad \Rightarrow \quad x = (x'\, y')^{1/4} \quad \text{and} \quad y = \left(\frac{x'}{y'}\right)^{1/4} .$$

(b)

$$\begin{aligned}
\frac{\partial}{\partial x'}\,\frac{\partial x'}{\partial x} &= \frac{\partial}{\partial x'}\, 2x\, y^2 = 2\,\frac{\partial}{\partial x'}\,(x'\, y')^{1/4}\left(\frac{x'}{y'}\right)^{1/2} = 2\,\frac{\partial}{\partial x'}\,\left(\frac{x'^3}{y'}\right)^{1/4} \\
&= \frac{1}{2}\left(\frac{x'^3}{y'}\right)^{-3/4} 3\,\frac{x'^2}{y'} = \frac{3}{2}\left(\frac{x^6\, y^6\, y^2}{x^2}\right)^{-3/4}\,\frac{x^4\, y^4\, y^2}{x^2} \\
&= \frac{3}{2}\,\left(x\, y^2\right)^{-3}\, x^2\, y^6 = \frac{3}{2x}\ . \\
\frac{\partial}{\partial x}\,\frac{\partial x'}{\partial x'} &= \frac{\partial}{\partial x}\, 1 = 0\, .
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\partial^2 x'}{\partial x\, \partial x}\,\frac{\partial x}{\partial x'} &= 2y^2\,\left(x' y'\right)^{-3/4}\,\frac{y'}{4} = \frac{y^2}{2}\,\left(x^4\right)^{-3/4}\,\frac{x^2}{y^2} = \frac{1}{2x} \\
\frac{\partial^2 x'}{\partial x\, \partial y}\,\frac{\partial y}{\partial x'} &= 4\, x\, y\,\left(\frac{x'}{y'}\right)^{-3/4}\left(\frac{1'}{4y'}\right) = x\, y\,\left(y^4\right)^{-3/4}\,\frac{y^2}{x^2} = \frac{1}{x}\, .
\end{aligned}$$

So the sum

$$\frac{\partial^2 x'}{\partial x\, \partial x}\,\frac{\partial x}{\partial x'} + \frac{\partial^2 x'}{\partial x\, \partial y}\,\frac{\partial y}{\partial x'} = \frac{3}{2x}$$

agrees with the first result of (b).

(d) The operators

$$\frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y}$$

commute because x and y are independent variables. To the contrary x and x' are not independet. So the order of the differentiations

$$\frac{\partial}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial x'}$$

matters.

(d) We have

$$p^{1'}_{1i}\, p^{i}_{1'} = \frac{\partial^2 x^{1'}}{\partial x^1\, \partial x^1}\, \frac{\partial x^1}{\partial x^{1'}} + \frac{\partial^2 x^{1'}}{\partial x^1\, \partial x^2}\, \frac{\partial x^2}{\partial x^{1'}} = \frac{3}{2x^1}$$

where the result follows from the previous calculation.