Special and General Relativity (PHZ 4601/5606) Fall 2018 Solutions

Set 12

29. Electromagnetic current conservation. Exercise 14.2 of Rindler.

We start off with Rindler (14.2), p.298,

$$E_{\mu\nu} = \Phi_{\nu:\mu} - \Phi_{\mu:\nu}\,,$$

where : denotes the covariant derivative. The field equation may the be written as

$$E_{\mu\nu:}{}^{\nu} = g^{\nu\hat{\nu}}\, E_{\mu\nu:\hat{\nu}} = g^{\nu\hat{\nu}}\, \Phi_{\nu:\mu\hat{\nu}} - g^{\nu\hat{\nu}}\, \Phi_{\mu:\nu\hat{\nu}} = \frac{4\pi}{c}\, J_\mu\,.$$

We want to show that the continuity equation

$$\begin{aligned} g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, E_{\mu\nu:\hat{\nu}\hat{\mu}} &= g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, \Phi_{\nu:\mu\hat{\nu}\hat{\mu}} - g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, \Phi_{\mu:\nu\hat{\nu}\hat{\mu}} \\ &= \frac{4\pi}{c}\, g^{\mu\hat{\mu}}\, J_{\mu:\hat{\mu}} = \frac{4\pi}{c}\, J_{\mu:}{}^{\mu} = 0\,. \end{aligned}$$

Here we have used that covariant differentiation commutes with contraction. See Rindler (10.34), p.212. Let us interchange the μ and ν dummies in the second term:

$$g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, \Phi_{\mu:\nu\hat{\nu}\hat{\mu}} = g^{\mu\hat{\mu}}\, g^{\nu\hat{\nu}}\, \Phi_{\nu:\mu\hat{\mu}\hat{\nu}}$$

After this we use Rindler (10.57), p.218 and get

$$\begin{aligned} \frac{4\pi}{c}\, J_{\mu:}{}^{\mu} &= g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, \left(\Phi_{\nu:\mu\hat{\nu}\hat{\mu}} - \Phi_{\nu:\mu\hat{\mu}\hat{\nu}}\right) = g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, \left(\Phi_{\alpha:\mu}\, R^{\alpha}{}_{\nu\hat{\nu}\hat{\mu}} + \Phi_{\nu:\alpha}\, R^{\alpha}{}_{\mu\hat{\nu}\hat{\mu}}\right) \\ &= g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, \left(\Phi_{\alpha:\mu}\, R_{\hat{\alpha}\nu\hat{\nu}\hat{\mu}} + \Phi_{\nu:\alpha}\, R_{\hat{\alpha}\mu\hat{\nu}\hat{\mu}}\right)\,. \end{aligned}$$

Re-labelling the dummies in the second term by $\alpha \to \beta$, $\nu \to \alpha$ and $\mu \to \nu$ gives

$$\frac{4\pi}{c}\, J_{\mu:}{}^{\mu} = g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, \Phi_{\alpha:\mu}\, R_{\hat{\alpha}\nu\hat{\nu}\hat{\mu}} + g^{\alpha\hat{\alpha}}\, g^{\nu\hat{\nu}}\, g^{\beta\hat{\beta}}\, \Phi_{\alpha:\beta}\, R_{\hat{\beta}\nu\hat{\alpha}\hat{\nu}}\,.$$

Next, replacing in the second term $\beta \to \mu$ gives

$$\frac{4\pi}{c}\, J_{\mu:}{}^{\mu} = g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, \Phi_{\alpha:\mu}\, \left(R_{\hat{\alpha}\nu\hat{\nu}\hat{\mu}} + R_{\hat{\mu}\nu\hat{\alpha}\hat{\nu}}\right)\,.$$

Using, respectively, identities (10.64), (10.62) of p.218 of Rindler, and relabelling we find

$$
\begin{gathered}
g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, R_{\hat{\mu}\nu\hat{\alpha}\hat{\nu}} = g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, R_{\hat{\alpha}\hat{\nu}\hat{\mu}\nu} = \\
-g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, R_{\hat{\alpha}\hat{\nu}\nu\hat{\mu}} = -g^{\nu\hat{\nu}}\, g^{\mu\hat{\mu}}\, g^{\alpha\hat{\alpha}}\, R_{\hat{\alpha}\nu\hat{\nu}\hat{\mu}}
\end{gathered}
$$

implying

$$
\frac{4\pi}{c}\, J_{\mu :}{}^{\mu} = 0\,.
$$