Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 14

35. Isotropic from of the Scharzschild metric.

Let us transform the factor $(1 - 2m/r)$ of the Schwarzschild metric by substituting

$$r = \left(1 + \frac{m}{2\overline{r}}\right)^2 \overline{r}\,.$$

One gets

$$1 - \frac{2m}{r} = 1 - 2m\,\left(1 + \frac{m}{2\overline{r}}\right)^{-2} \frac{1}{\overline{r}} = \frac{(1+m/2\overline{r})^2\,\overline{r} - 2m}{(1+m/2\overline{r})^2\,\overline{r}}$$
$$= \frac{(1+m/\overline{r}+m^2/4\overline{r}^2)\,\overline{r} - 2m}{(1-m/2\overline{r})^2\,\overline{r}} = \frac{(1-m/2\overline{r})^2\,\overline{r}}{(1+m/2\overline{r})^2\,\overline{r}} = \frac{(1-m/2\overline{r})^2}{(1+m/2\overline{r})^2}\,.$$

Therefore, we have the new coefficient of dt^2.

To transform dr^2 we calculate

$$\frac{dr}{d\overline{r}} = 2\,\left(1 + \frac{m}{2\overline{r}}\right)\,\left(-\frac{m}{2\overline{r}^2}\right)\,\overline{r} + \left(1 + \frac{m}{2\overline{r}}\right)^2$$
$$= \left(1 + \frac{m}{2\overline{r}}\right)\,\left(-\frac{m}{\overline{r}^2} + 1 + \frac{m}{2\overline{r}}\right) \;=\; \left(1 + \frac{m}{2\overline{r}}\right)\,\left(1 - \frac{m}{2\overline{r}}\right)$$

and find

$$\left(1 - \frac{2m}{r}\right)^{-1} dr^2 = \left(1 + \frac{m}{2\overline{r}}\right)^4 d\overline{r}^2$$

From substituint r^2 the factor $(1 + m/2\overline{r})^4\,\overline{r}^2$ applies to the solid angle, so that our result is

$$d\vec{s}^{\,2} = \frac{(1-m/2\overline{r})^2}{(1+m/2\overline{r})^2}\,dt^2 - \left(1 + \frac{m}{2\overline{r}}\right)^4\,\left[d\overline{r}^2 + \overline{r}^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right]\,.$$

Cartesian coordinates are then defined in the usual way,

$$\overline{z} = \overline{r}\,\cos\theta\,,\quad \overline{x} = \overline{r}\,\sin\theta\,\cos\phi\,,\quad \overline{y} = \overline{r}\,\sin\theta\,\sin\phi\,,$$

and our final result is

$$d\vec{s}^{\,2} = \frac{(1-m/2\overline{r})^2}{(1+m/2\overline{r})^2}\,dt^2 - \left(1 + \frac{m}{2\overline{r}}\right)^4\,\left(d\overline{x}^2 + d\overline{y}^2 + d\overline{z}^2\right)\,.$$

Asymptotic behavior: $r = \overline{r}$ for $\overline{r} \to \infty$.
Schwarzschild radius (singularity): $r = 2m$, $\overline{r} = m/2$.