Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 12

37. Radial coordinate velocity of light in Schwarzschild space-time. Exercise 11.2 of Rindler.

As $\theta,\ \phi = const$,

$$d\vec{s}^{\,2} = \alpha\, dt^2 - \alpha^{-1}\, dr^2 \quad \text{with} \quad \alpha = 1 - 2m/r\,.$$

Light: $d\vec{s}^{\,2} = 0 \ \Rightarrow\ dt = \pm\alpha^{-1}\, dr$. So the coordinate velocity is

$$\frac{dr}{dt} = \pm\alpha \ \to 0 \ \text{ for } \ r \to 2m\,.$$

For the coordinate time to a mirror at r_1 from $r_0 > r_1$ and back, we have

$$\begin{aligned}\triangle t &= 2\int_{r_1}^{r_0} (1-2m/r)^{-1} dr = 2\int_{r_1}^{r_0} r/(r-2m)\, dr \\ &= 2\int_{r_1}^{r_0} (r-2m+2m)/(r-2m)\, dr = 2\int_{r_1}^{r_0} \left[1 + 2m/(r-2m)\right]\, dr \\ &= 2\left[r_0 - r_1 + 2m\, \ln\left(\frac{r_0-2m}{r_1-2m}\right)\right] \ \to\ \infty \ \text{ for } \ r_1 \to 2m\,.\end{aligned}$$

As we watch a source fall towards the horizon, it appears to slow down and to stop at the horizon. The times observed with standard clocks are

$$\triangle\tau_0 = \sqrt{1 - \frac{2m}{r_0}}\,\triangle t \quad \to\ \infty \text{ for } \ r_1 \to 2m$$

and

$$\triangle\tau_1 = \sqrt{1 - \frac{2m}{r_1}}\,\triangle t \quad \to\ 0 \text{ for } \ r_1 \to 2m\,.$$

By Eq. (11.25) of Rindler the redshift of the light received by us approaches infinity:

$$\frac{\lambda_0}{\lambda_1} = \left(\frac{1-2m/r_0}{1-2m/r_1}\right) \ \to\ \infty \ \text{ for } \ r_1 \to 2m\,.$$

So the signal becomes ever dimmer, since both the frequency of photons reaching us becomes smaller, and the rate at which they arrive also decreases, each by the Doppler factor.