Covariant Differentiation

In the following we use the notation of Rindler (7.1), p.132 for partial derivatives:

$$p^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}\,, \quad p^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}\,, \quad p^{i'}_{ij} = \frac{\partial^2 x^{i'}}{\partial x^i\,\partial x^j}\,. \tag{1}$$

Note that $p^{i'}_{ik}\,p^k_{k'}$ should be used for

$$\frac{\partial}{\partial x^{k'}}\,p^{i'}_i\,,$$

because in $p^{i'}_{ik'}$ **the order** of the x^i and $x^{k'}$ differentiations would matter, while i and k commute in $p^{i'}_{ik}\,p^k_{k'}$ (see exercise 23).

Definition:

$$F^i{}_{j;k} = F^{i'}{}_{j',k'}\,p^i_{i'}\,p^{j'}_j\,p^{k'}_k\,. \tag{2}$$

Here $_{;k}$ indicates covariant differentiation, $_{,k'}$ is ordinary differentiation. The LHS is in an arbitrary coordinate system $\{x^i\}$ and the RHS in a geodesic (freely falling) coordinate system $\{x^{i'}\}$. Let us re-write the derivative $F^{i'}{}_{j',k'}$ of the RHS in terms of 2-tensors in the $\{x^i\}$ frame:

$$\begin{aligned} F^{i'}{}_{j',k'} &= \frac{\partial}{\partial x^{k'}}\,F^{i'}{}_{j'} = \frac{\partial}{\partial x^{k'}}\left(F^l{}_m\,p^{i'}_l\,p^m_{j'}\right) \\ &= F^l{}_{m,n}\,p^{i'}_l\,p^m_{j'}\,p^n_{k'} + F^l{}_m\,p^{i'}_{l\,n}\,p^n_{k'}\,p^m_{j'} + F^l{}_m\,p^{i'}_l\,p^m_{j'\,k'}\,. \end{aligned} \tag{3}$$

Now we work out the contractions of the p factors in (3) with the $p^i_{i'}\,p^{j'}_j\,p^{k'}_k$ factor in (2):

$$p^{i'}_l\,p^m_{j'}\,p^n_{k'}\;p^i_{i'}\,p^{j'}_j\,p^{k'}_k = \delta^i_l\,\delta^m_j\,\delta^n_k \tag{4}$$

$$p^{i'}_{l\,n}\,p^n_{k'}\,p^m_{j'}\;p^i_{i'}\,p^{j'}_j\,p^{k'}_k = p^i_{i'}\,p^{i'}_{l\,n}\,\delta^n_k\,\delta^m_j = \Gamma^i{}_{l\,k}\,\delta^m_j \tag{5}$$

$$p^{i'}_l\,p^m_{j'\,k'}\;p^i_{i'}\,p^{j'}_j\,p^{k'}_k = p^m_{j'\,k'}\,p^{j'}_j\,p^{k'}_k\,\delta^i_l = -\Gamma^m{}_{j\,k}\,\delta^i_l\,. \tag{6}$$

Carrying out the δ contractions we find:

$$\begin{aligned} F^i{}_{j;k} = F^{i'}{}_{j',k'}\,p^i_{i'}\,p^{j'}_j\,p^{k'}_k &= F^l{}_{m,n}\,\delta^i_l\,\delta^m_j\,\delta^n_k + F^l{}_m\,\Gamma^i{}_{l\,k}\,\delta^m_j - F^l{}_m\,\Gamma^m{}_{j\,k}\,\delta^i_l \\ &= F^i{}_{j,k} + F^l{}_j\,\Gamma^i{}_{l\,k} - F^i{}_m\,\Gamma^m{}_{j\,k} \end{aligned} \tag{7}$$

in agreement with Rindler (10.28), p.211. Notably, the auxiliary geodesic system has been eliminated. Application of Eq. (7) are given in the following.

Scalar:

$$\Phi_{;i} = \Phi_{,i} \,. \tag{8}$$

Contravariant Vector:

$$V^i{}_{;j} = V^i{}_{,j} + V^a\,\Gamma^i{}_{aj} \,. \tag{9}$$

Covariant Vector:

$$V_{i;j} = V_{i,j} - V_a\,\Gamma^a{}_{ij} \,. \tag{10}$$

Contravariant rank two tensor:

$$F^{ij}{}_{;k} = F^{ij}{}_{,k} + F^{aj}\,\Gamma^i{}_{ak} + F^{ia}\,\Gamma^j{}_{ak} \,. \tag{11}$$

Covariant rank two tensor:

$$F_{ij;k} = F_{ij,k} - F_{aj}\,\Gamma^a{}_{ik} - F_{ia}\,\Gamma^a{}_{jk} \,. \tag{12}$$

Contravariant rank three tensor:

$$T^{i_1 i_2 i_3}{}_{;j} = T^{i_1 i_2 i_3}{}_{,j} + T^{a i_2 i_3}\,\Gamma^{i_1}{}_{aj} + T^{i_1 a i_3}\,\Gamma^{i_2}{}_{aj} + T^{i_1 i_2 a}\,\Gamma^{i_3}{}_{aj} \tag{13}$$

Covariant rank three tensor:

$$T_{i_1 i_2 i_3;j} = T_{i_1 i_2 i_3,j} - T_{a i_2 i_3}\,\Gamma^a{}_{i_1 j} - T_{i_1 a i_3}\,\Gamma^a{}_{i_2 j} - T_{i_1 i_2 a}\,\Gamma^a{}_{i_3 j} \,. \tag{14}$$

For mixed tensors the positive and negative contributions are simply added up, e.g.,

$$T^{i_1 i_2 i_3}_{j_1 j_2 j_3;k} = T^{i_1 i_2 i_3}_{j_1 j_2 j_3,k} + T^{a i_2 i_3}_{j_1 j_2 j_3}\,\Gamma^{i_1}{}_{ak} + T^{i_1 a i_3}_{j_1 j_2 j_3}\,\Gamma^{i_2}{}_{ak} + T^{i_1 i_2 a}_{j_1 j_2 j_3}\,\Gamma^{i_3}{}_{ak} \tag{15}$$

$$- T^{i_1 i_2 i_3}_{a j_2 j_3}\,\Gamma^a{}_{j_1 k} - T^{i_1 i_2 i_3}_{j_1 a j_3}\,\Gamma^a{}_{j_2 k} - T^{i_1 i_2 i_3}_{j_1 j_2 a}\,\Gamma^a{}_{j_3 k} \,. \tag{16}$$

The order of contravariant indices with respect to one another matters as does the order of covariant indices with respect to one another, while the relative order of contravariant to covariant indices is not relevant for the contraction rules (but matters of course for the tensor itself). Generalization to rank n tensors should now be obvious.