Riemann Curvature Tensor

In the following we use the notation colon : instead of semi-colon ; for the covariant derivative. Covariant derivatives do **not commute**. As in Rindler before Eq. (10.55) on p.217 we define

$$\begin{bmatrix} h \\ j \end{bmatrix} = V^h_{:j} = V^h_{,j} + V^a\,\Gamma^h_{aj}\ . \tag{1}$$

Then we have

$$V^h_{:jk} = \begin{bmatrix} h \\ j \end{bmatrix}_{:k} = \begin{bmatrix} h \\ j \end{bmatrix}_{,k} + \begin{bmatrix} b \\ j \end{bmatrix}\Gamma^h_{bk} - \begin{bmatrix} h \\ b \end{bmatrix}\Gamma^b_{jk}\,, \tag{2}$$

$$V^h_{:kj} = \begin{bmatrix} h \\ k \end{bmatrix}_{:j} = \begin{bmatrix} h \\ k \end{bmatrix}_{,j} + \begin{bmatrix} b \\ k \end{bmatrix}\Gamma^h_{bj} - \begin{bmatrix} h \\ b \end{bmatrix}\Gamma^b_{kj}\,, \tag{3}$$

for the two orders of the covariant derivatives. Using the symmetry $\Gamma^b_{jk} = \Gamma^b_{kj}$ the difference becomes

$$V^h_{:jk} - V^h_{:kj} = \begin{bmatrix} h \\ j \end{bmatrix}_{,k} + \begin{bmatrix} b \\ j \end{bmatrix}\Gamma^h_{bk} - \begin{bmatrix} h \\ k \end{bmatrix}_{,j} - \begin{bmatrix} b \\ k \end{bmatrix}\Gamma^h_{bj}\ .$$

Comparing corresponding terms we get

$$\begin{aligned} \begin{bmatrix} h \\ j \end{bmatrix}_{,k} - \begin{bmatrix} h \\ k \end{bmatrix}_{,j} &= V^h_{,jk} + V^a_{,k}\,\Gamma^h_{aj} + V^a\,\Gamma^h_{aj,k} \\ &- V^h_{,kj} - V^a_{,j}\,\Gamma^h_{ak} - V^a\,\Gamma^h_{ak,j} \end{aligned}$$

where the product rule has been applied. Using now the symmetry $V^h_{,jk} = V^h_{,kj}$ we get

$$\begin{bmatrix} h \\ j \end{bmatrix}_{,k} - \begin{bmatrix} h \\ k \end{bmatrix}_{,j} = V^a_{,k}\,\Gamma^h_{aj} + V^a\,\Gamma^h_{aj,k} - V^a_{,j}\,\Gamma^h_{ak} - V^a\,\Gamma^h_{ak,j}\,. \tag{4}$$

Next, inserting the definitions of the brackets

$$\begin{bmatrix} b \\ j \end{bmatrix}\Gamma^h_{bk} - \begin{bmatrix} b \\ k \end{bmatrix}\Gamma^h_{bj} = V^b_{,j}\,\Gamma^h_{bk} + V^a\,\Gamma^b_{aj}\,\Gamma^h_{bk} - V^b_{,k}\,\Gamma^h_{bj} - V^a\,\Gamma^b_{ak}\,\Gamma^h_{bj} \tag{5}$$

holds. Combining from (4) and (5)

$$V^a_{,k}\,\Gamma^h_{aj} + V^b_{,j}\,\Gamma^h_{bk} - V^a_{,j}\,\Gamma^h_{ak} - V^b_{,k}\,\Gamma^h_{bj} = 0$$

is seen to be true due to symmetry under exchanging j and k. Left over are from (4) $V^a \, \Gamma^h_{aj,k} - V^a \, \Gamma^h_{ak,j}$ and from (5) $V^a \, \Gamma^b_{aj} \, \Gamma^h_{bk} - V^a \, \Gamma^b_{ak} \, \Gamma^h_{bj}$. So, the result is

$$\begin{aligned} V^h_{:jk} - V^h_{:kj} &= V^a \, \Gamma^h_{aj,k} + V^a \, \Gamma^b_{aj} \, \Gamma^h_{bk} - V^a \, \Gamma^h_{ak,j} - V^a \, \Gamma^b_{ak} \, \Gamma^h_{bj} \\ &= -V^a \, R^h{}_{ajk} \, , \end{aligned} \tag{6}$$

where

$$R^h{}_{ijk} = \Gamma^h_{ij,k} + \Gamma^b_{ij} \, \Gamma^h_{bk} - \Gamma^h_{ik,j} - \Gamma^b_{ik} \, \Gamma^h_{bj} \tag{7}$$

defines the Riemann curvature tensor. These are 256 functions, which are related by many symmetries (identities). See Rindler p.218/219.