Let us consider the geodesic between two given points A and B, and a bundle of neighboring curves also connecting A to B. Our problem is to find the curve satisfying the variational principle

$$0 = \delta \int \mathrm{d}s = \delta \int \left|g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right|^{1/2} = \delta \int_{u_1}^{u_2} \left|g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}u}\frac{\mathrm{d}x^j}{\mathrm{d}u}\right|^{1/2} \mathrm{d}u, \tag{10.6}$$

where in the last integral u is an arbitrary parameter which continuously parametrizes the entire bundle of comparison curves (much like ct in Fig. 10.1) so as to have fixed values u_1 and u_2 at the fixed end-points of the curves. With $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}u$, eqn (10.6) becomes

$$\delta \int \left|g_{ij}\dot{x}^i\dot{x}^j\right|^{1/2} \mathrm{d}u =: \delta \int L(x^i, \dot{x}^i)\,\mathrm{d}u = 0. \tag{10.7}$$

The reader is perhaps familiar from classical mechanics with this kind of 'variational' problem. Its solution $x^i = x^i(u)$ is found by integrating the well-known Euler–Lagrange differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0. \tag{10.8}$$

At this stage we may conveniently take u to be the arc s *along the solution curve*, provided that curve is not null. This makes $L = 1$ along the solution curve and allows us to replace the awkward Lagrangian L defined by eqn (10.7) (the square root of a metric is never pleasant) by essentially its square,

$$\mathcal{L} := g_{ij}\dot{x}^i\dot{x}^j = \pm L^2. \tag{10.9}$$

For consider the variational principle

$$\delta \int \mathcal{L}\,\mathrm{d}s = 0, \tag{10.10}$$

$2L\frac{d}{ds}\frac{\partial L}{\partial \dot{x}^i}$

whose solution is determined by the N equations //

$$\mathbb{L}_i := \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0 = \frac{\mathrm{d}}{\mathrm{d}s}\left(2L\frac{\partial L}{\partial \dot{x}^i}\right) - 2L\frac{\partial L}{\partial x^i}, \tag{10.11}$$

where, for future reference, we have introduced the notation $\mathbb{L}_i$ for the LHS of the ith equation. Since the solution satisfies $L = \text{const}$, we see that the Euler–Lagrange equations for $\mathcal{L}$ are equivalent to those for L. They are, in fact, the standard equations used in the practical determination of geodesics.

Mainly for theoretical purposes, we now further examine the structure of the set of eqns (10.11)(i). With (10.9) substituted it becomes, successively (with a few index tricks),

$$\begin{aligned}
\mathbb{L}_i &\equiv \frac{\mathrm{d}}{\mathrm{d}s}(2g_{ij}\dot{x}^j) - g_{jk,i}\dot{x}^j\dot{x}^k = 0 && (1)\\
&2g_{ij,k}\dot{x}^j\dot{x}^k + 2g_{ij}\ddot{x}^j - g_{jk,i}\dot{x}^j\dot{x}^k = 0 \qquad (10.12) && (2)\\
&(g_{ij,k} + g_{ik,j} - g_{jk,i})\dot{x}^j\dot{x}^k + 2g_{ij}\ddot{x}^j = 0. && (3)
\end{aligned}$$

So if we now define the so-called *Christoffel symbols of the first kind* by

$$\Gamma_{ijk} := \tfrac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}), \tag{10.13}$$

and those *of the second kind* (also called *connection coefficients*) by raising the index i,

$$\Gamma^i_{jk} := g^{hi}\Gamma_{hjk}, \tag{10.14}$$

we can re-express eqns (10.12) (after raising i and recalling $g^i{}_j = \delta^i_j$) as

Einstein Eq.

$$\tfrac{1}{2}L^i \equiv \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{10.15}$$

This is the alternative standard form of the set of differential equations for geodesics. It shows that geodesics are fully determined by an initial point O and an initial direction $\dot{x}^i_0$. [At least, if we assume analyticity: for eqn (10.15) yields $\ddot{x}^i_0$, (10.15) differentiated yields $\dddot{x}^i_0$, etc., and so the Taylor series for $x^i(u)$ can be constructed.]

The Christoffel symbols play an important role in differential geometry and also in GR. But *they are not tensors*! We note their symmetry:

$$\Gamma_{ijk} = \Gamma_{ikj}, \qquad \Gamma^i_{jk} = \Gamma^i_{kj}, \tag{10.16}$$

and the 'inverse' of eqn (10.13), which shows they are not tensors:

$$g_{ij,k} = \Gamma_{ijk} + \Gamma_{jik}. \tag{10.17}$$

[For proof, just substitute (10.13) into the RHS.] This latter equation, together with (10.13), shows that the vanishing of all the Γs at one point is equivalent to the vanishing of the derivatives of all the gs.

For the actual calculation of the Γs of a given metric (an often unavoidable task in GR) one can sometimes bypass (10.13) and simply compare the coefficients of $\dot{x}^j\dot{x}^k$ in eqns (10.15) with those in the written-out versions of eqns (10.11). This works best in the case of orthogonal coordinates when eqns (10.11) and (10.15) differ only by a factor $2g_{ii}$. But then one can also use the formulae of the Appendix of this book. (See, for example, Exercise 10.3.)

The reader may worry that eqn (10.15) 'does not know' that the parameter is supposed to be the arc. But, essentially, it knows: one can show [cf. after (10.42)] that *any solution* $x^i(u)$ *of* (10.15) *with* $\cdot = \mathrm{d}/\mathrm{d}u$ *necessarily satisfies*

$$g_{ij}\dot{x}^i\dot{x}^j = \text{const}, \tag{10.18}$$

that is, $L = \text{const}$, which was the basic assumption of our derivation.

We can re-write eqn (10.18) in the form $\mathbf{ds}^2 = (\text{const})\,\mathrm{d}u^2$ and deduce, first, that the sign of $\mathbf{ds}^2$ is necessarily constant along a geodesic, and second, that every solution parameter is 'affinely' related to the arc:

$$u = as + b. \tag{10.19}$$

systems.] For proof we have, with $\dot{} \equiv \mathrm{d}/\mathrm{d}u$,

$$\dot{x}^{i'} = p^{i'}_j \dot{x}^j$$

$$\ddot{x}^{i'} = p^{i'}_{jk}\dot{x}^j\dot{x}^k + p^{i'}_i \ddot{x}^i. \tag{10.24}$$

Now any geodesic $x^i(u)$ through P referred to an affine parameter must satisfy eqn (10.15), and thus $\ddot{x}^i = 0$ at P if $\{x^i\}$ is a geodesic system. For two such systems we then deduce from (10.24) that $(p^{i'}_{jk})_P = 0$. Conversely, if $(p^{i'}_{jk})_P = 0$ and $\{x^i\}$ is geodesic at P, every geodesic in $\{x^{i'}\}$ also satisfies $\ddot{x}^{i'} = 0$ at P and so, by (10.15), $(\Gamma^{i'}_{j'k'})_P = 0$.

Next suppose we transform from a system $\{x^{i'}\}$ that is geodesic at P to an arbitrary system $\{x^i\}$. Then all geodesics through P, *at* P satisfy $\ddot{x}^{i'} = 0$ and thus, 'flipping' $p^{i'}_i$ in eqn (10.24),

$$\ddot{x}^i + p^{i'}_{jk} p^i_{i'} \dot{x}^j \dot{x}^k = 0.$$

Comparison with (10.15) then shows that *at* P:

$$\Gamma^i_{jk} = p^{i'}_{jk} p^i_{i'}. \tag{10.25}$$

However, if we differentiate the relation $p^i_{i'} p^{i'}_j = \delta^i_j$ [cf. (7.2)] with respect to x^k, we find

$$p^i_{i'} p^{i'}_{jk} + p^i_{i'k'} p^{k'}_k p^{i'}_j = 0,$$

which, after a cosmetic dummy replacement, yields the following alternative to the RHS of (10.25):

$$\Gamma^i_{jk} = -p^i_{j'k'} p^{j'}_j p^{k'}_k. \tag{10.26}$$

Both versions will be needed in the next section.

~~Any coordinate system in terms of which the geodesics through a given point P have~~ equations like (10.22) are called *Riemannian*. If additionally they are orthonormal (that is, pseudo-Euclidean) at P, they are called *normal coordinates* with pole at P. Any two normal systems at P are related to each other globally by generalized rotations about P; that is, transformations that preserve the (pseudo-)Euclidean metric *at* P. (In spacetime these are the LTs.) For, $u \propto s$, so $\mathrm{d}s/\mathrm{d}u = s/u$ and therefore

$$(g_{ij})_P\, x^i(Q)\, x^j(Q) = (g_{ij})_P \frac{\mathrm{d}x^i}{\mathrm{d}u}\frac{\mathrm{d}x^j}{\mathrm{d}u} u^2 = s^2 = (g_{i'j'})_P\, x^{i'}(Q)\, x^{j'}(Q),$$

where $(g_{ij})_P$ and $(g_{i'j'})_P$ both represent the (pseudo-)Euclidean metric at P. That ~~proves it. For example, $\sum(x^i)^2 = \sum(x^{i'})^2$ if V^N is properly Riemannian. These~~

Since the 3-index ps (in the last two terms) vanish between geodesic systems, $F^i_{j,k}$ is seen to behave tensorially between such systems. We now define the *covariant derivative* $F^i_{j;k}$ of F^i_j, in an arbitrary coordinate system $\{x^i\}$, as the tensor transform of the partial derivative in a geodesic system $\{x^{i'}\}$:

$$F^i_{j;k} = F^{i'}_{j',k'} p^i_{i'} p^{j'}_j p^{k'}_k .$$

Since the partial derivatives are related tensorially among geodesic systems, $F^i_{j;k}$ is related tensorially to *all* of them, by transitivity (cf. Section 7.2C). The same is true of $F^{i''}_{j'';k''}$, similarly defined in another arbitrary system $\{x^{i''}\}$. So, again by transitivity and symmetry, $F^i_{j;k}$ and $F^{i''}_{j'';k''}$ are tensorially related, which shows that our definition indeed creates a tensor.

In eqn (10.27), let us now assume the system $\{x^{i'}\}$ to be geodesic and the system $\{x^i\}$ to be arbitrary; let us write a for the dummy j in the last term, and similarly for the dummy i in the term before; if we then 'flip' $p^{i'}_i$, $p^j_{j'}$, and $p^k_{k'}$ [cf. (7.3)], we get

$$F^{i'}_{j',k'} p^i_{i'} p^{j'}_j p^{k'}_k = F^i_{j,k} + F^a_j p^{i'}_{ak} p^i_{i'} + F^i_a p^a_{j'k'} p^{j'}_j p^{k'}_k .$$

The LHS is the required covariant derivative $F^i_{j;k}$. On the RHS we can eliminate all reference to the auxiliary geodesic coordinate system by applying our carefully prepared formulae (10.25) and (10.26). Thus we finally obtain (with some relief!)

$$F^i_{j;k} = F^i_{j,k} + F^a_j \Gamma^i_{ak} - F^i_a \Gamma^a_{jk}. \tag{10.28}$$

This formula is typical. For the general tensor field $F^{\cdots}_{\cdots}$ it again begins with the partial derivative, followed by *positive* Γ-terms resulting from the replacement, one after the other, of all the contravariant indices of $F^{\cdots}_{\cdots}$ by a dummy which is linked to a Γ that also takes over the replaced free index; similarly there is a *negative* Γ-term for each covariant index. It is an easy pattern to remember. In particular, for a scalar $\phi(x^i)$ it gives

$$\phi_{;i} = \phi_{,i}, \tag{10.29}$$

which is not surprising, since $\phi_{,i}$ is a tensor.

A pleasant property of the covariant derivative is that it satisfies the ordinary rules for differentiating sums and (inner and outer) products:

$$(S^{\cdots}_{\cdots} + T^{\cdots}_{\cdots})_{;k} = S^{\cdots}_{\cdots;k} + T^{\cdots}_{\cdots;k}, \tag{10.30}$$

$$(S^{\cdots}_{\cdots} T^{\cdots}_{\cdots})_{;k} = S^{\cdots}_{\cdots;k} T^{\cdots}_{\cdots} + S^{\cdots}_{\cdots} T^{\cdots}_{\cdots;k}. \tag{10.31}$$

For, *at the pole of geodesic coordinates, where covariant and partial differentiation are the same*, the above equations are trivially true; but, being tensor equations, they must then be true in all coordinates.

~~[For (iii), consider $(\mathbf{V}\cdot\mathbf{W})^* = \dot{\mathbf{V}}\cdot\mathbf{W} + \mathbf{V}\cdot\dot{\mathbf{W}}$]. When C is a geodesic, $\mathbf{N} = 0$ and the RHS of (10.53) vanishes; FW transport then reduces to parallel transport.~~

10.5 The Riemann curvature tensor

Let us introduce the following notation for the repeated covariant derivative of a tensor:

$$(T^{\cdots}_{\cdots;i})_{;j} = T^{\cdots}_{\cdots;ij},$$

and similarly for higher derivatives. (This parallels the notation we have already introduced in Section 7.2E for repeated *partial* derivatives: $T^{\cdots}_{\cdots,ij}$.) From elementary calculus we are familiar with the commutativity of partial derivatives: $T^{\cdots}_{\cdots,ij} = T^{\cdots}_{\cdots,ji}$. But the corresponding statement for covariant derivatives is generally false. Only in *flat* space is it true that

$$T^{\cdots}_{\cdots;ij} = T^{\cdots}_{\cdots;ji} \quad \text{(flat space)}. \tag{10.54}$$

For then we can always choose (pseudo-)Euclidean coordinates ($g_{ij} = \pm\delta^i_j$), in which the Γs vanish globally, and in which even repeated covariant differentiation therefore reduces to repeated partial differentiation; in these coordinates, (10.54) is true, and being tensorial, it must then be true in all coordinates. In the general case we cannot prove (10.54) by going to the pole of geodesic coordinates: for while the Γs vanish there, the same is not true of their derivatives. And it is these which cause the inequality.

Let us do the calculation for the simplest case, a vector V^h. We have

$$V^h{}_{;j} = \left[V^h{}_{,j} + V^a\Gamma^h_{aj}\right] =: \begin{bmatrix} h \\ j \end{bmatrix}, \text{ say}$$

$$V^h{}_{;jk} = \begin{bmatrix} h \\ j \end{bmatrix}_{,k} + \begin{bmatrix} b \\ j \end{bmatrix}\Gamma^h_{bk} - \begin{bmatrix} h \\ b \end{bmatrix}\Gamma^b_{jk}.$$

If we write this out in full, reverse j, k and subtract, we find

$$V^h{}_{;jk} - V^h{}_{;kj} = -V^a R^h{}_{ajk}, \tag{10.55}$$

where

$$R^h{}_{ijk} = \Gamma^h_{ik,j} - \Gamma^h_{ij,k} + \Gamma^h_{aj}\Gamma^a_{ik} - \Gamma^h_{ak}\Gamma^a_{ij}. \tag{10.56}$$

Since V^a in (10.55) is an arbitrary vector, it follows from the quotient rule (cf. Exercise 7.2) that $R^h{}_{ijk}$ must be a tensor. It is, in fact, one of the most important tensors in Riemannian geometry, the so-called *Riemann curvature tensor*. Its relation to the curvature at a given point will become apparent a little later. In flat space it clearly vanishes. And conversely, its global vanishing can be shown to imply flat space.