Special and General Relativity (PHZ 4601/560)

Final (two pages) December 13, 10-12 am.

1. **Special Relativity: Electromagnetic invariants (30%).**

   In the notation of Rindler the components of the electromagnetic field tensor are given by

$$\left(E^{\alpha\beta}\right) = \begin{pmatrix} 0 & -b^3 & b^2 & e^1 \\ b^3 & 0 & -b^1 & e^2 \\ -b^2 & b^1 & 0 & e^3 \\ -e^1 & -e^2 & -e^3 & 0 \end{pmatrix} .$$

   (a) Calculate $E_{\alpha\beta}\, E^{\alpha\beta}$ as an expression of $\vec{b}$ and $\vec{e}$.

   (b) Write down the matrix of the dual tensor

$$^{*}E_{\alpha\beta} = \frac{1}{2}\, \epsilon_{\alpha\beta\gamma\delta}\, E^{\gamma\delta}$$

   in terms of the components of $\vec{b}$ and $\vec{e}$. Here $\epsilon_{\alpha\beta\gamma\delta}$ is the totally anti-symmetric Levi-Civita tensor, $\epsilon^{1234} = +1$.

   (c) Calculate $^{*}E_{\alpha\beta}\, E^{\alpha\beta}$ as an expression of $\vec{b}$ and $\vec{e}$.

2. **Schwarzschild effective potential (40%).**

   The Schwarzschild effective potential is given by

$$V_{GR}(r) = \frac{h^2}{r^2} - \frac{2m_s}{r} - \frac{2m_s\, h^2}{r^3} ,$$

   where h is the specific angular momentum (i.e., angular momentum per rest mass).

   In gravitational units $c = G = 1$ the average distance of the earth to the sun becomes approximately 10^8 solar masses m_s, $r_e = 10^8\,[m_s]$, and a second s becomes approximately $s = 2 \times 10^5\,[m_s]$.

   (a) For the orbit of the earth about the sun: Give a rough estimate of h in units of the solar mass m_s.

   (b) Continue with $h = 10^4\,[m_s]$ and find the extrema of the effective Schwarzschild potential. Give the numerical values of $V_{\min} = V_{GR}(r_+)$, $V_{\max} = V_{GR}(r_-)$, and $r_\pm$.

(c) Sketch $V_{GR}(r)$ in the neighbourhood of its minimum value $V_{\min} = V_{GR}(r_+)$. Label the axis and indicate a few tics (≥ 3) that have to be quantitatively correct.

(d) Sketch $V_{GR}(r)$ in the neighbourhood of its maximum value $V_{\max} = V_{GR}(r_-)$. Label the axis as in (c).

3. **Schwarzschild radius and density (10%).**

Consider a sphere of radius R and uniform mass density ρ. Find the smallest radius R_0 for which you can proof that $r_S > R$ holds for $R > R_0$, where r_S is the Schwarzschild radius. Proof it by calculating $r_s - R$ explicitely.

4. **Einstein's field equations (20%).**

(a) Transform the equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -\kappa\, T_{\mu\nu}$$

into an equation for $R_{\mu\nu}$ as function of $T_{\mu\nu}$.

(b) Consider the Einstein field equation with a cosmological constant in 4 dimensions (3 space, 1 time):

$$G_{\mu\nu} + g_{\mu\nu}\,\Lambda = -\kappa\, T_{\mu\nu}\,.$$

Show that $R = n\,\Lambda$ holds in vacuum and find n.