Fall 2018 Special and General Relativity (PHZ 4601/560)

Final Solutions

1. **Special Relativity: Electromagnetic invariants.**

   (a)

$$\left(E^{\alpha\beta}\right) = \begin{pmatrix} 0 & -b^3 & b^2 & e^1 \\ b^3 & 0 & -b^1 & e^2 \\ -b^2 & b^1 & 0 & e^3 \\ -e^1 & -e^2 & -e^3 & 0 \end{pmatrix} \quad \Rightarrow \quad (E_{\alpha\beta}) = \begin{pmatrix} 0 & -b^3 & b^2 & -e^1 \\ b^3 & 0 & -b^1 & -e^2 \\ -b^2 & b^1 & 0 & -e^3 \\ +e^1 & +e^2 & +e^3 & 0 \end{pmatrix} .$$

   Therefore,

$$E_{\alpha\beta}\, E^{\alpha\beta} = 2\left(\vec{b}^2 - \vec{e}^2\right) \, .$$

   (b) $E_{12} = \epsilon_{1234}\, E^{34} = -E^{34} = +e^3, \qquad E_{13} = \epsilon_{1324}\, E^{24} = +E^{24} = -e^1,$
   $E_{13} = \epsilon_{1324}\, E^{24} = +E^{24} = -e^1$, and so on implies

$$({}^*E_{\alpha\beta}) = \begin{pmatrix} 0 & -e^3 & e^2 & b^1 \\ e^3 & 0 & -e^1 & b^2 \\ -e^2 & e^1 & 0 & b^3 \\ -b^1 & -b^2 & -b^3 & 0 \end{pmatrix} .$$

   (c) Therefore,

$${}^*E_{\alpha\beta}\, E^{\alpha\beta} = 4\, \vec{b} \cdot \vec{e} \, .$$

2. **Schwarzschild effective potential:**

   (a) We have $h = r_e\, v_e$ where r_e is the average distance of the earth from the sun and v_e the average velocity of the earth perpendicular to r_e. Approximating the orbit of the earth by a circle,

$$v_e = \frac{2\pi\, r_e}{T} = 0.9962 \times 10^{-4} \, ,$$

   where the period T is one earth year and we use gravitational units with the mass of the sun as unit mass. Multiplying with r_e, we obtain

$$h = 9962\, [m_s] \, .$$

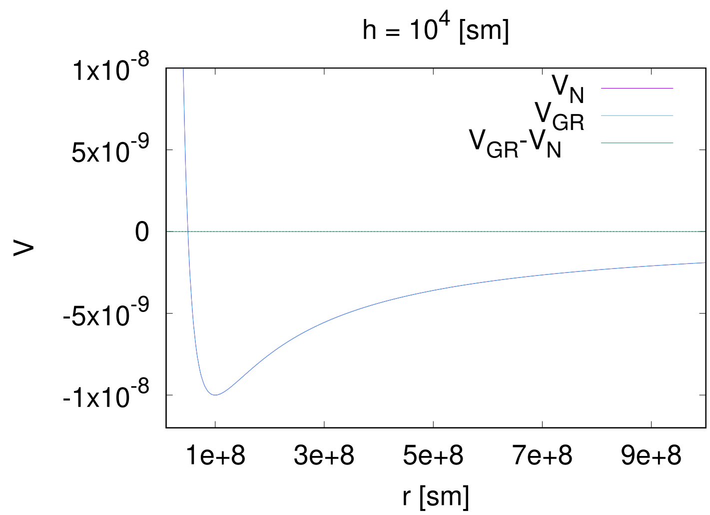
Second solution for (a): The values of h can also be found from the minimum of $V_{GR}(r)$, which corresponds to the orbit of a circle. Then

$$0 = V'_{GR}(r) = \frac{d}{dr} V_{GR}(r) = -\frac{2\,h^2}{r^3} + \frac{2}{r^2} + \frac{6\,h^2}{r^4} \quad \Rightarrow \quad h^2 = \left(\frac{2}{r^2}\right) / \left(\frac{2}{r^3} - \frac{6}{r^4}\right) ,$$

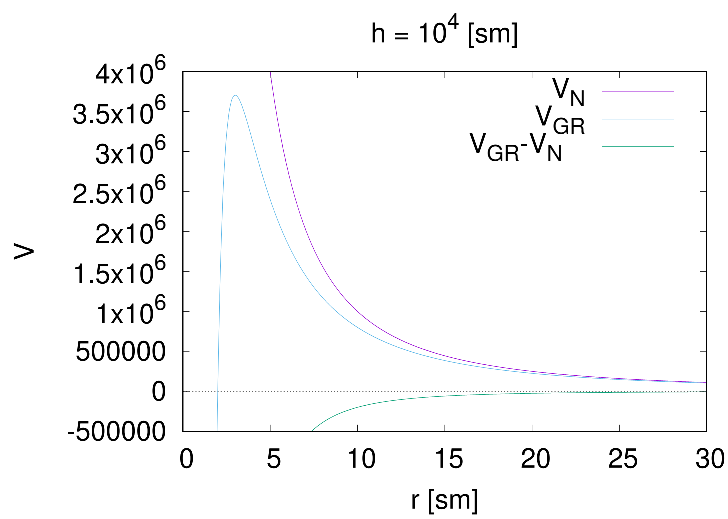
and with $r = r_e = 10^8\,[m_s]$ we get

$$h \approx r_e = 10^4\,[m_s]\,.$$

The difference between the two results comes from the approximation for T.



Newton's classical region.



Schwarzschild singularity region.

(b) The extrema of the Schwarzschild effective potential follow from

$$0 = V'_{GR}(r) = \frac{d}{dr} V_{GR}(r) = -\frac{2\,h^2}{r^3} + \frac{2}{r^2} + \frac{6\,h^2}{r^4} \,,$$

where we used $m_s = 1$. Multiplying with r^4 gives

$$0 = r^2 - h^2\, r + 3\, h^2$$

with the solutions

$$r_\pm = \frac{h^2}{2} \pm \sqrt{\left(\frac{h^2}{2}\right)^2 - 3h^2}\,,$$

where $V_{GR}^{\rm max} = V_{GR}(r_-)$ and $V_{GR}^{\rm min} = V_{GR}(r_+)$. Continueing with $h = 10^4\,[m_s]$, we find

$$r_- = 3\,[m_s] \quad \text{and} \quad r_+ = 10^8\,[m_s] \quad \text{with} \quad V_{GR}^{\rm max} = 3.704 \times 10^6 \quad \text{and} \quad V_{GR}^{\rm min} = -10^{-8}\,.$$

(c) See the first figure (Newton's effective potential V_N is included, but was not asked).

(d) See the second figure (Newton's effective potential V_N is included, but was not asked).

3. **Schwarzschild radius and density (10%).**

The Scharzschild radius is

$$r_s = 2m = \frac{8\pi}{3}\, r^3\, \rho \;\Rightarrow\; R_0 = \frac{8\pi}{3}\, R_0^3\, \rho \;\Rightarrow\; R_0^2 = \frac{3}{8\pi\,\rho}$$

and $r_S > R$ for $R > R_0$, because r_s increases $\sim R^3$, i.e., faster than R. This rules out a minimum at $r_s = R_0$:

$$r_s - R = \frac{8\pi}{3}\,\rho\, R^3 - R = \frac{8\pi}{3}\,\rho\, R\left(R^2 - \frac{3}{8\pi\,\rho}\right) = \frac{8\pi}{3}\,\rho\, R\left(R^2 - R_0^2\right)\,.$$

4. **Einstein's equations.**

(a) We have

$$R^\mu{}_\mu - \frac{1}{2}\, g^\mu{}_\mu\, R = -\kappa\, T^\mu{}_\mu \;\Rightarrow\; R - 2\,R = -\kappa\, T \;\Rightarrow\; R = \kappa\, T$$

and, therefore,

$$R_{\mu\nu} - \kappa\,\frac{1}{2}\, g_{\mu\nu}\, T = -\kappa\, T_{\mu\nu} \;\Rightarrow\; R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T\right)\,.$$

(b) In vacuum we have

$$R^\mu{}_\mu - \frac{1}{2}\, g^\mu{}_\mu\, R + g^\mu{}_\mu\, \Lambda = 0 \;\Rightarrow\; R - 2\,R + 4\,\Lambda = 0 \;\Rightarrow\; R = 4\,\Lambda\,,$$

i.e., $n = 4$.