

Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 10

33. Isotropic form of the Schwarzschild metric.

Let us transform the crucial factor $(1 - 2m/r)$ of the Schwarzschild metric by substituting $r = (1 + m/2\bar{r})^2 \bar{r}$:

$$\begin{aligned} 1 - \frac{2m}{r} &= 1 - 2m \left(1 + \frac{m}{2\bar{r}}\right)^{-2} \frac{1}{\bar{r}} = \frac{(1 + m/2\bar{r})^2 \bar{r} - 2m}{(1 + m/2\bar{r})^2 \bar{r}} \\ &= \frac{(1 + m/\bar{r} + m^2/4\bar{r}^2) \bar{r} - 2m}{(1 + m/2\bar{r})^2 \bar{r}} = \frac{(1 - m/2\bar{r})^2 \bar{r}}{(1 + m/2\bar{r})^2 \bar{r}} = \frac{(1 - m/2\bar{r})^2}{(1 + m/2\bar{r})^2} \end{aligned}$$

Therefore, we have the new coefficient of dt^2 . To transform dr^2 we calculate

$$\frac{dr}{d\bar{r}} = 2 \left(1 + \frac{m}{2\bar{r}}\right) \left(-\frac{m}{2\bar{r}^2}\right) \bar{r} + \left(1 + \frac{m}{2\bar{r}}\right)^2 = \left(1 + \frac{m}{2\bar{r}}\right)^2 \left(1 - \frac{m}{2\bar{r}}\right)^2$$

and find

$$\left(1 - \frac{2m}{r}\right)^{-1} dr^2 = \left(1 + \frac{m}{2\bar{r}}\right)^4 d\bar{r}^2$$

From substituting r^2 the factor $(1 + m/2\bar{r})^4 \bar{r}^2$ applies to the solid angle, so that our result is

$$d\bar{s}^2 = \frac{(1 - m/2\bar{r})^2}{(1 + m/2\bar{r})^2} dt^2 - \left(1 + \frac{m}{2\bar{r}}\right)^4 \left[d\bar{r}^2 + \bar{r}^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2) \right].$$

Cartesian coordinates are then defined in the usual way,

$$\bar{z} = \bar{r} \cos \bar{\theta}, \quad \bar{x} = \bar{r} \sin \bar{\theta} \cos \bar{\phi}, \quad \bar{y} = \bar{r} \sin \bar{\theta} \sin \bar{\phi},$$

and our final result is

$$d\bar{s}^2 = \frac{(1 + m/2\bar{r})^2}{(1 + m/2\bar{r})^2} dt^2 - \left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2).$$