Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 9

30. Use the appendix, p.419, 420, of Rindler to show Eq. (11.4).

We choose the identification ($c = 1$)

$$\begin{aligned} d\vec{s}^{\,2} &= a\,(dx^1)^2 + b\,(dx^2)^2 + (dx^3)^2 + (dx^4)^2 \\ &= a\,(dt)^2 + b\,(dr)^2 + c\,(d\theta)^2 + d\,(d\phi)^2 \end{aligned}$$

with $a = \exp[+A(r)]$, $b = -\exp[+B(r)]$, $c = -r^2$, $d = -r^2 \sin^2\theta$. The non-zero derivatives are the $' = \partial/\partial x^2 = d/dr$ and on d only $\partial/\partial x^3 = \partial/\partial\theta$. Therefore, the relevant contribution to the Ricci tensor are

$$R_{tt} + R_{11} = \beta\, a_{22} - \beta\, a_2\,(\alpha\, a_2 + \beta\, b_2 - \gamma\, c_2 - \delta_2\, d_2)\,,$$

where α, β, γ, δ are defined by $\alpha = 1/2a$, $\beta = 1/2b$, $\gamma = 1/2c$, $\delta = 1/2d$. We have

$$a' = A' \exp(+A)\,, \quad b' = -B' \exp(+B) \quad c' = -2r \quad d' = -2r\,\sin^2\theta$$

and $a'' = A'' \exp(A) + (A')^2 \exp(A)$. Inserting these gives

$$\begin{aligned} R_{tt} &= -\exp(A-B)\,\frac{1}{2}\,\left(A'' + A'^2\right) \\ &\quad + \exp(A-B)\,\frac{1}{2}\,A'\,\left(\frac{1}{2}\,A' + \frac{1}{2}\,B' - \frac{1}{r} - \frac{1}{r}\right) \\ &= -\exp(A-B)\,\left(\frac{1}{2}\,A'' - \frac{1}{4}\,A'\,B' + \frac{1}{4}\,A'^2 + A'/r\right)\,. \end{aligned}$$