Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 10

33. Isotropic from of the Scharzschild metric.

Let us transform the crucial factor $(1-2m/r)$ of the Schwarzschild metric by substituting $r=(1+m/2\overline{r})^2\,\overline{r}$:

$$1-\frac{2m}{r}=1-2m\left(1+\frac{m}{2\overline{r}}\right)^{-2}\frac{1}{\overline{r}}=\frac{(1+m/2\overline{r})^2\,\overline{r}-2m}{(1+m/2\overline{r})^2\,\overline{r}}$$

$$=\frac{(1+m/\overline{r}+m^2/4\overline{r}^2)\,\overline{r}-2m}{(1-m/2\overline{r})^2\,\overline{r}}=\frac{(1-m/2\overline{r})^2\,\overline{r}}{(1+m/2\overline{r})^2\,\overline{r}}=\frac{(1-m/2\overline{r})^2}{(1+m/2\overline{r})^2}$$

Therefore, we have the new coefficient of dt^2. To transform dr^2 we calculate

$$\frac{dr}{d\overline{r}}=2\left(1+\frac{m}{2\overline{r}}\right)\left(-\frac{m}{2\overline{r}^2}\right)\overline{r}+\left(1+\frac{m}{2\overline{r}}\right)^2=\left(1+\frac{m}{2\overline{r}}\right)\left(1-\frac{m}{2\overline{r}}\right)$$

and find

$$\left(1-\frac{2m}{r}\right)^{-1}dr^2=\left(1+\frac{m}{2\overline{r}}\right)^4 d\overline{r}^2$$

From substituint r^2 the factor $(1+m/2\overline{r})^4\,\overline{r}^2$ applies to the solid angle, so that our result is

$$d\vec{s}^{\,2}=\frac{(1-m/2\overline{r})^2}{(1+m/2\overline{r})^2}\,dt^2-\left(1+\frac{m}{2\overline{r}}\right)^4\left[d\overline{r}^2+\overline{r}^2\,(d\theta^2+\sin^2\theta\,d\phi^2)\right]\,.$$

Cartesian coordinates are then defined in the usual way,

$$\overline{z}=\overline{r}\,\cos theta\,,\quad \overline{x}=\overline{r}\,\sin\theta\,\cos\phi\,,\quad \overline{y}=\overline{r}\,\sin\theta\,\sin\phi\,,$$

and our final result is

$$d\vec{s}^{\,2}=\frac{(1+m/2\overline{r})^2}{(1+m/2\overline{r})^2}\,dt^2-\left(1+\frac{m}{2\overline{r}}\right)^4\left(d\overline{x}^2+d\overline{y}^2+d\overline{z}^2\right)\,.$$