Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 12

38. Retarded potential and Lorenz gauge.

In the retarded solution

$$\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\int d^3y\,\frac{J^\mu(x^0-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}$$

for the potential we replace x^ν by $x^\nu+\epsilon^\nu$

$$\Phi^\mu(\{x^\nu+\epsilon^\nu\}) = \frac{1}{c}\int d^3y\,\frac{J^\mu(x^0+\epsilon^0-|\vec{x}+\vec{\epsilon}-\vec{y}|,\vec{y})}{|\vec{x}+\vec{\epsilon}-\vec{y}|}$$

and expand to order ϵ on the left hand side:

$$\Phi^\mu(\{x^\nu\}) + \epsilon^\rho\,\frac{\partial}{\partial x^\rho}\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\int d^3y\,\frac{J^\mu(x^0+\epsilon^0-|\vec{x}+\vec{\epsilon}-\vec{y}|,\vec{y})}{|\vec{x}+\vec{\epsilon}-\vec{y}|}\,.$$

On the right hand side we substitute $\vec{y}=\vec{y}\,'+\vec{\epsilon}$ and get

$$\Phi^\mu(\{x^\nu\}) + \epsilon^\rho\,\frac{\partial}{x^\rho}\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\int d^3y'\,\frac{J^\mu(x^0+\epsilon^0-|\vec{x}-\vec{y}\,'|,\vec{y}\,'+\vec{\epsilon})}{|\vec{x}-\vec{y}\,'|}\,,$$

where we used that the Jacobian is one, i.e., $d^3y = d^3y'$. We now expand to order ϵ on the right hand side (note $y'^0+\epsilon^0 = x^0+\epsilon^0-|\vec{x}-\vec{y})$ and obtain

$$\epsilon^\rho\,\frac{\partial}{x^\rho}\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\epsilon^\rho\int\frac{d^3y'}{|\vec{x}-\vec{y}\,'|}\,\frac{\partial}{\partial y'^\rho}\,J^\mu(y'^0,\vec{y}\,)$$

As the ϵ^ρ are arbitrary, we can drop them, and we can also eliminate the prime on the integrations variable. So,

$$\frac{\partial}{x^\rho}\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\int\frac{d^3y}{|\vec{x}-\vec{y}\,|}\,\frac{\partial}{\partial y^\rho}\,J^\mu(\{y^\nu\})$$

holds. Contracting σ with μ

$$\frac{\partial}{x^\mu}\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\int\frac{d^3y}{|\vec{x}-\vec{y}\,|}\,\frac{\partial}{\partial y^\mu}\,J^\mu(\{y^\nu\}) = 0$$

follows due to current conservation on the right hand side, and the Lorenz gauge $\partial_\mu\Phi^\mu = 0$ is obtained on the left hand side.