Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 5

16. End of Spacetrip.

We use energy-momentum conservation

$$0 = dp = \begin{pmatrix} dp_r^0 \\ dp_r^1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} dp_e^0 \\ dp_e^1 \\ 0 \\ 0 \end{pmatrix} ,$$

where the subscript r stands for rocket and e for exhaust. In the temporary rest frame of the rocket we have

$$dp_e^1 = -v\, dm \,, \qquad dp_r^1 = m\, du = m\, g\, d\tau \,,$$

where v is the velocity of the exhaust, dm the infinitesimal change of rest mass of the rocket (not the exhaust!) and du the infinitesimal velocity of the rocket. Note that the exhaust may have no rest mass (v=c). The equation for dp_e^1 (no γ in front of v!) follows from

$$dp_e^0 = -dp_r^0 = c\, dm \;\; \text{and} \;\; -\beta = \frac{dp_e^1}{dp_e^0} : \quad dp_e^1 = -\beta\, dp_e^0 = v\, dm \,.$$

With this definition $\beta = v/c$ is positive, because dp_e^1 is negative when we choose dp_r^1 positive. Now, $0 = -v\, dm + g\, d\tau$ and separation of variables gives

$$\frac{dm}{m} = -\frac{g}{v}\, d\tau \;\Rightarrow\; \int_{m_0}^{m(\tau)} \frac{dm}{m} = -\frac{g}{v} \int_0^{\tau} d\tau' \;\Rightarrow\; \ln\left(\frac{m(\tau)}{m_0}\right) = -\frac{g\,\tau}{v} \,.$$

(1) The mass of the spaceship decreases with its proper time according to

$$m(\tau) = m_0 \, \exp\left(-\frac{g\tau}{v}\right) .$$

(2) With $v = c$ and after a trip of twenty years the remaining fraction of the mass is $m(20\,\text{years})/m_0 = 1.10 \times 10^{-9}$.