Special and General Relativity (PHZ 4601/5606) Fall 2017
Classwork and Homework

Every exercise counts 10 points unless stated differently.

Set 1:

(1) Homework, due F 9/1/2017 before class. In 2D Euclidean space a straight line goes in K from $\begin{pmatrix}1\\1\end{pmatrix}$ to $\begin{pmatrix}2\\2\end{pmatrix}$. From where to where does it go in K' which is rotated by $\phi = \pi/8$? Is it still a straight line?

(2) Homework, due F 9/1/2017 before class. In 2D Euclidean space find the matrix that takes $\begin{pmatrix}x'\\y'\end{pmatrix}$ in K' to $\begin{pmatrix}x\\y\end{pmatrix}$ in K along the lines used in the lecture from K to K' (orientation of the frames the same as there).

Set 2:

(3) Homework, due M 9/18/2017 before class. Exercise 1.6 of Rindler.

(4) Homework, due M 9/18/2017 before class. Use $c = 3 \times 10^5$ $[km/s]$, $g = 9.8$ $[m/s^2]$, $[ft] = 0.3[m]$, and one year = 365 days. Estimate the time difference encountered due to the gravitational frequency shift after running for one year identical atomic clocks positioned in Colorado at an altitude of $5400\,[ft]$ and at the Greenwich Observatory at an altitude of $80\,[ft]$. State the result in microsecond and round to the first two digits. Assume that g is constant in the range from $80\,[ft]$ to $5400\,[ft]$ and give a reason why that is a reasonable approximation.

(5) Classwork, due W 9/5/2017 in class (late acceptance 9/18/2017). Use $c = 3 \times 10^5$ $[km/s]$ and $g = 9.81$ $[m/s^2]$, and one year = 365 days to calculate g in units of light years $[ly]$ and years $[y]$ to three significant digits.

(6) Homework, due M 9/18/2017 before class. Exercise 1.9 of Rindler. By "source" a light source is meant.

Set 3:

(7) Homework, due F 9/22/2017 before class: Spherical coordinates.

(1) Given are the spherical coordinates

$$\theta = 3\,\pi/4\ [rad],\ \phi = \frac{4\,\pi}{3}\ [rad] \ \text{ and } \ r = 5\ [ly]\,.$$

Calculate the corresponding (contravariant) Cartesian coordinates x^1, x^2 and x^3.

(2) Given are

$$x^1 = 2\ [ly],\ x^2 = -1\ [ly] \ \text{ and } \ x^3 = -3\ [ly]\,.$$

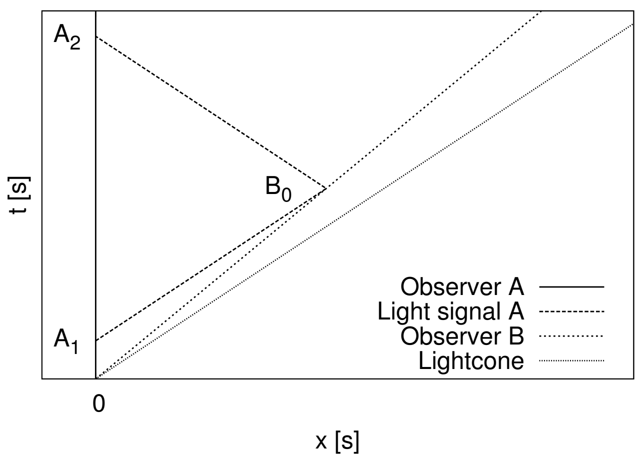
Find the corresponding polar coordinates θ, ϕ and r (angles in $[rad]$).

(8) Homework, due F 9/22/2017 before class: Euclidean and hyperbolic rotations.
Consider the 2D Euclidean rotation

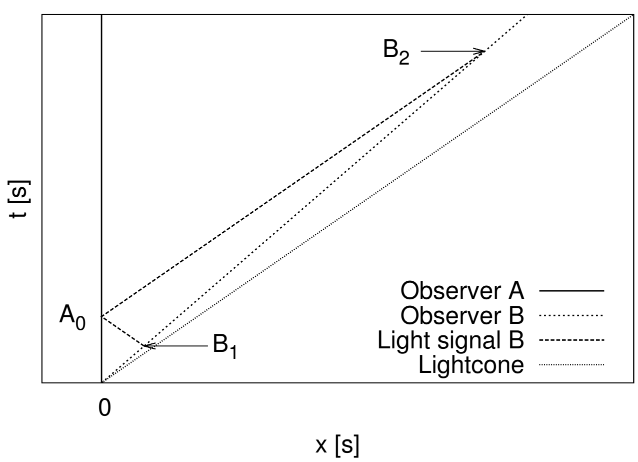
$$\begin{pmatrix} x'^{\,1} \\ x'^{\,4} \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x^1 \\ x^4 \end{pmatrix}$$

and substitute $\phi = i\,\zeta$, $x^4 = i\,x^0$, $x'^{\,4} = i\,x'^{\,0}$. Write out the equations for $x'^{\,1}$ and $x'^{\,0}$ in terms of sinh and cosh.

(9) Classwork, due M 9/18/2017 in class: Time and relativistic distance measurements (synchronization of clocks).
A Cesium clock counts 2,757,789,531,312 cycles (starting at 0). Find the elapsed time in seconds and round to the nearest integer number.
An observer O_1 is located in an inertial system and flashes at times 1, 2, 3 … $[s]$ light signals towards another observer O_2, who reflects them back with a mirror. Assume that the returned signals are received by O_1 at the following times:
(1) 1.002, 2.002, 3.002 … $[s]$,
(2) 1.002, 2.004, 3.006 … $[s]$,
(3) 1.002, 2.004, 3.008 … $[s]$.
Determine for each case whether the data are consistent with assuming that O_2 is also in an inertial system. If this is the case, write down the equation for the distance of O_2 as function of the time as seen by O_1. Approximate the speed of light by $300,000\,[km/s]$, but perform all calculations with a precision of at least four digits.



Minkowski space in which observer A is at rest and flashes a light signal at observer B, who moves with speed $4c/5$ and flashes the signal back.



Minkowski space in which observer A is at rest and observer B moves with speed $4c/5$. Observer B flashes a light signal at observer A, who flashes it back.

Set 4:

(10) Homework, due F 9/29/2017 before class. Time in Minkowski space.

Use natural units with $c = 1$ for this exercise.
In the figures the Minkowski space is parametrized by the coordinates

of the rest frame of an observer A. While observer A stays at rest, observer B moves with speed $4c/5$ along the positive x axis. At their common origin both, A and B, have set their clocks to zero. After 15 $[s]$ observer A emits at position A_1 a light signal which reaches observer B at position B_0, who flashes it back at observer A, who receives it at position A_2 as drawn in the first figure.

1. What are the coordinates of the position B_0?
2. What is the time on the clock of observer B at position B_0?
3. What is the time on the clock of observer A when the back signal is received at position A_2?

In the second figure observer B flashes a light signal at A from position B_1 when the time on the clock of B shows 15 $[s]$. It is received by A at position A_0 and reflected back at B, who receives it at position B_2.

4. What are the coordinates of the position B_1?
5. What time does the clock of A show at position A_0?
6. What are the coordinates of B_2 in the Minkowski space and what time does the clock of B show at this position?
7. Compare the time reported by A at position A_2 with the time reported by B at position B_2. Without the actual calculation are there reasons why these times should agree or disagree?

(11) Homework, due F 9/29/2017 before class. Exercise 2.9 of Rindler.

(12) Homework, due F 9/29/2017 before class. Time dilation for a satellite.

The circular orbit of a satellite is described by its radius R and angular velocity ω. Use special relativity to calculate the ratio of the proper time interval $d\tau$ of a clock on the satellite over the time interval dt of the inertial frame in which the center of the orbit is at rest (neglect gravitational effects).
Use $d\tau$ for a clock on the satellite and dt for a clock fixed on the equator of the earth. Calculate the numerical value $1-d\tau/dt$ for a satellite that orbits above the equator at an altitude of 160 [km]. Compare with the competing effect from gravity.

(13) Graduate students only: Homework, due F 9/29/2017 before class. Addition theorem for transverse velocity components.

The addition theorem of velocities can also be derived as follows: Assume that IF K' moves with respect to K with velocity $\vec{v} = v\,\hat{x}^1$ and a particle moves with respect to K' with velocity $\vec{u}^{\,\prime}$:

$$x'^{\,i} = c^{-1} u'^{\,i} x'^{\,0}.$$

The Lorentz boost equations

$$\begin{aligned} x'^{\,0} &= x^0 \cosh(\zeta) - x^1 \sinh(\zeta)\,, \\ x'^{\,1} &= -x^0 \sinh(\zeta) + x^1 \cosh(\zeta)\,, \\ x'^{\,i} &= x^i,\ (i = 2, 3)\,. \end{aligned}$$

imply

$$\gamma\,(x^1 - \beta x^0) = c^{-1} u'^{\,1} \gamma\,(x^0 - \beta x^1)\,.$$

Sorting with respect to x^1 and x^0 gives

$$\gamma\,\left(1 + \frac{u'^{\,1} v}{c^2}\right)\,x^1 = c^{-1}\gamma\,(u'^1 + v)\,x^0\,.$$

Using the definition of the velocity in K, $\vec{x} = c^{-1}\vec{u}\,x^0$, gives

$$u^1 = c\,\frac{x^1}{x^0} = \frac{u'^1 + v}{1 + u'^1 v/c^2}\,.$$

Derive along similar lines the equation for the two other components $u^i, (i = 2, 3)$.

Set 5:

(14) Homework, due F 10/6/2017 before class.

Two particles, each of mass m and with negligible kinetic energy, annihilate into a particle of mass M and a photon.

1. Use natural units with $c = 1$ and calculate the photon energy $E_\gamma = |p_\gamma|$ as function of M in the range $0 \le M < 2m$.
2. Compute $E_\gamma(M)$ for $M = 0\,,\ 0.5\,m,\ 1\,m,\ \sqrt{2}\,m\,,\ \sqrt{3}\,m.$
3. Sketch $f(M) = 2m - M$ and $E_\gamma(M)$ in one graph together.

(15) Homework, due F 10/6/2017 before class. Spacetrip and elapsed time on Earth.

Assume that the earth is in an inertial frame. A spaceship leaves the earth in the year zero. The spaceship is constructed so that it has an acceleration g in its own frame (to make the occupants feel comfortable). By its own clock, it accelerates on a straight-line path for 5 years, decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. What is the time on earth? (This problem is adapted from Jackson, Electrodynamics.)
Instructions: Use $g = 9.81$ $[m/s^2]$, one year $= 365 \times 24 \times 3600$ $[s]$, and for the speed of light $c = 300,000$ $[km/s]$. Calculate three significant digits. Hint: Find first $\zeta(\tau)$, where τ is the proper time in the rocket and $\zeta(\tau)$ its rapidity with respect to Earth.

(16) Homework, due F 10/6/2017 before class. End of spacetrip.

Assume that the spaceship of the previous exercise moves by exhausting particles (photons) at the speed of light c.

1. Derive an expression for $m(\tau)$, the (remaining) mass of the spaceship at proper time τ.

2. Which fraction of the original mass is left, after the spacetrip has been performed?

(17) Classwork, due M 10/2/2017 in class, 5 points.
An astrophysical observation.

For light from some galaxy the spectrum

$$\lambda = (729.2\,[nm])\, m^2/(m^2-4)\,, \quad m = 3,\, 4,\, 5,\, \dots$$

is observed. Find the speed at which the galaxy moves away or towards us (ignore the possibility of transverse motion and the expansion of space). Note: In quantum mechanics books you find for the Balmer spectrum of the hydrogen atom $\lambda = (364.56\,[nm])\, m^2/(m^2-4)$.

Set 6:

(18) Homework, due F 10/13/2017 before class. Relative velocity.

An observer in an inertial frame S' measures the velocities of two particles and finds

$$\vec{\beta}'_1 = \begin{pmatrix} \beta'^1_1 \\ \beta'^2_1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

along the x direction for the first particle and

$$\vec{\beta}'_2 = \begin{pmatrix} \beta'^1_2 \\ \beta'^2_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

along the y direction for the second particle.

Find the speed of the second particle in the rest frame of the first particle. You may use the formulas of problem 13.

(19) Homework, due F 10/13/2017 before class.
The four-dimensional Levi-Civita tensor $\epsilon^{\alpha\beta\gamma\delta}$ is completely antisymmetric in its indices and let $\epsilon^{1234} = 1$. Show the following properties:
(1) Show $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$.
(2) Express $\epsilon_{\alpha\beta\gamma_1\delta_1}\epsilon^{\alpha\beta\gamma_2\delta_2}$ in terms of Kronecker delta.
(3) Express $\epsilon_{\alpha\beta_1\gamma_1\delta_1}\epsilon^{\alpha\beta_2\gamma_2\delta_2}$ in terms of Kronecker delta.

(20) Graduate students only: Homework, due F 10/13/2017 before class.

The dual tensor is defined by

$${}^*E^{\alpha\beta} = \frac{1}{2}\,\epsilon^{\alpha\beta\gamma\delta}E_{\gamma\delta}\,.$$

Use Eq. (7.32) of the book to express the dual tensor in terms of $\vec{e}$ and $\vec{b}$. Subsequently, compare ${}^*E^{\alpha\beta}{}_{,\alpha} = 0$ with the homogeneous Maxwell equations, Eq. (7.26) and (7.27) of the book.

(21) Classwork, due 10/6/2016 in class. Electromagnetic field tensor.

Let $E^{\alpha\beta}$ be an antisymmetric tensor of rank 2. Compare

$$\frac{\partial E^{\alpha\beta}}{\partial\, x^\alpha} = \partial_\alpha E^{\alpha\beta} = E^{\alpha\beta}{}_{,\alpha} = \frac{4\pi}{c}\,J^\beta$$

for $\beta = 4$ with the inhomogeneous Maxwell equations

$$\nabla\cdot\vec{e} = 4\pi\rho = \frac{4\pi}{c}\,J^4, \quad \nabla\times\vec{b} - \frac{1}{c}\frac{\partial\vec{e}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

to identify E^{44}, E^{14}, E^{24} and E^{34} in terms of $\vec{e}$ and $\vec{b}$. Continue with $\beta = 1$, then $\beta = 2$. Do you need $\beta = 3$ too to determine all components of the tensor $E^{\alpha\beta}$ in terms of $\vec{e}$ and $\vec{b}$?

Write down the matrix of the finally obtained electromagnetic field tensor $\left(E^{\alpha\beta}\right)$ in terms of $\vec{e}$ and $\vec{b}$.

Prepare for the Midterm on F October 20!

Set 7:

(22) Classwork, due 10/16/2016. Twin travel in 2D Minkowski space.

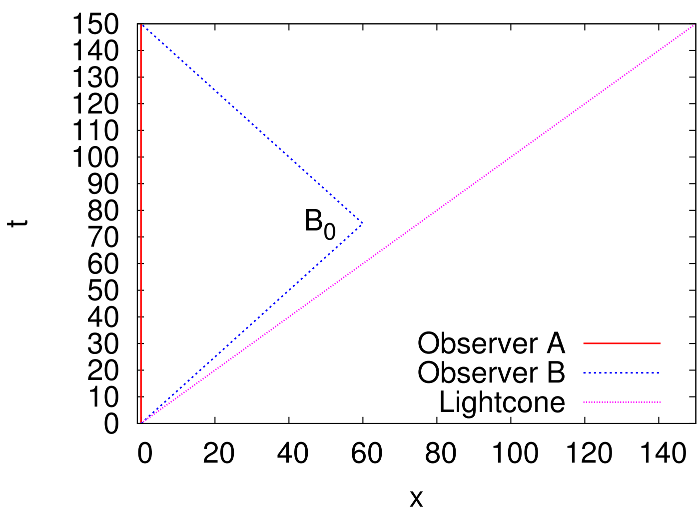


Figure: Minkowski space in which observer B travels away from observer A to the space-time point B_0 and back to A. Natural units with $c = 1$ and arbitrary time units are used.

The figure above is drawn in the rest frame of observer A. The initial position of observers A as well as B is $\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Observer B travels in the rest frame K of A with constant velocity β_1 away from A, turns at position $\begin{pmatrix} 75 \\ 60 \end{pmatrix}$ and travels back with velocity $-\beta_1$ to meet A again.

(a) What is the value of β_1?

(b) Consider the inertial frame K' in which B is for the first part of its travel at rest, which we call "initial rest frame (IRF)". The IRF is arranged so that its origin agrees at time zero with the origin of the rest frame of A. At what time leaves B the $x = 0$ axis of the IRF frame?

(c) Find the coordinates of the last position of B (where B meets A again) in the IRF and sketch the entire travel in this frame.

(d) Calculate the velocity for the second part of the travel of B in the IRF K'.

(23) Homework, due F 10/27/2017 before class. Distance in a saddle.

Consider a saddle described by the function

$$z = f(x, y) = x^2 - y^2$$

in arbitrary units. A two-dimensional being walks a distance $r = 10$ from the origin $x = y = 0$ of the saddle into its $\hat{x}$ direction. At which value of x is it? Give at least three significant digits. You may want to use

http://www.wolframalpha.com/calculators/integral-calculator/

for the integration.

(24) Homework, due F 10/27/2017 before class. Rindler 9.8.

Hints: First, complete the square for dt. Then, apply the transformation $t' = t + f(x, y)$ (called a "gauge transformation" in the context of gravity) and find a function $f(x, y)$ so that the metric becomes static (i.e., has no cross terms $dx\,dt$ and so on).

(25) Graduate students only: Homework, due F 10/27/2017 before class.

Derive the Euler-Lagrange equations from the action principle

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0\,, \quad (i = 1, \ldots, N)\,.$$

Prepare to present you solution in class.

Set 8:

(26) Homework, due F 11/3/2017 before class. Consider a flat 2D plane mapped by the standard polar coordinates.

(a) Write down ds^2.
(b) What are the coefficients g_{ij}?
(c) What are the coefficients g^{ij}?
(d) Calculate the Christoffel symbols for this system.
(e) Using

$$\dot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0$$

derive the two coupled, second-order differential equations that describe the motion of a free particle in this system.

(27) Homework, due F 11/3/2017 before class. Verify that the differential equations of the previous problem are fulfilled by

$$r = \sqrt{y_0^2 + s^2} \quad \text{and} \quad \tan(\theta) = \frac{y_0}{s}\,, \quad \text{where} \quad \cdot = \frac{d}{ds}$$

and y_0 is a constant. Describe the motion in the x-y plane.

(28) Graduate students only: Homework, due F 11/3/2017 before class. In the following units with $c = 1$ are used. Solve the spaceship problem (15) by transforming $(d\beta)_S = g\,d\tau$ to $(\beta)_E$, where E is the earth frame, considered to be inertial, and S is the instantaneous rest frame of the spaceship at velocity $\beta(\tau)$ seen from earth. You may use Mathematica for the integral over $(d\beta)_E$.

Set 9:

(29) Homework, due W 11/8/2017 before class. Consider the spacetime metric $d\vec{s}^{\,2} = dt^2 - dl^2$, where dl^2 is an arbitrary and possibly time-dependent metric. Are coordinate lines with t variable and the other coordinates fixed at constant values geodesic? Hint: Investigate whether the geodesic equations of the Lagrangian

$$L = \dot{t}^{\,2} - g_{ij}\,\dot{x}^i\,\dot{x}^j$$

allow for the proposed solutions.

(30) Homework, due W 11/8/2017 before class. Use the appendix of Rindler, p.419, 420, to show Eq. (11.4) for the Ricci tensor element R_{tt}.

Set 10: Three problems.

(31) Homework, due F 11/17/2017 before class. Gravitational Units. In the following use $G = 6.674 \times 10^{-11}\,[m^3\,kg^{-1}\,s^{-2}]$ for the gravitational constant, $m_e = 5.972 \times 10^{24}\,[kg]$ for the mass of the earth, which is considered to be a perfect sphere of radius $r_e = 6371 \times 10^3\,[m]$ of uniform mass density. Perform calculations for this problem to the three leading digits (rounding the fourth digit).

(a) Calculate the Scharzschild radius r_e^s of the earth in units of meter $[m]$ and, subsequently, the ratio r_e^s/r_e.
Continue using gravitational units defined by $c = G = 1$.

(b) Write down the values for m_e and r_e in units of seconds [s]. Subsequently, calculate the ratio $2\,m_e/r_e$.

(c) Write down the values for m_e and r_e in units of years [y]. Subsequently, calculate the ratio $2\,m_e/r_e$.

(d) Write down the values for m_e and r_e in units of meters [m]. Subsequently, calculate the ratio $2\,m_e/r_e$.

(e) Write down the values for m_e and r_e in units of light years [ly]. Subsequently, calculate the ratio $2\,m_e/r_e$.

(32) Homework, due F 11/17/2017 before class.
Solar system: Schwarzschild radius, ruler distance and radar distance. In the following use $G = 6.674 \times 10^{-11}\,[m^3\,kg^{-1}\,s^{-2}]$ for the gravitational constant, $m_s = 1.9886 \times 10^{30}\,[kg]$ for the mass of the sun, which is considered to be a perfect sphere of radius $r_s = 6957 \times 10^5\,[m]$ of uniform mass density. Assume $r_1 = 149.6 \times 10^9\,[m]$ for the coordinate distance of the earth to the center of the sun. Neglect the mass of the earth (or assume it to be replaced by a satellite). Perform calculations for this problem to the three leading digits (rounding the fourth digit).

(a) Calculate the Scharzschild radius r_s^s of the sun and, subsequently, the ratio r_s^s/r_s.

(b) Calculate the ruler (geodesic) distance from the earth to the surface of the sun and compare it with the corresponding coordinate distance.

(c) Calculate the radar distance from the earth to the surface of the sun and compare it with the corresponding coordinate and ruler distances.

(33) Homework, due F 11/17/2017 before class. Isotropic form of the Schwarzschild metric. Substitute

$$r = \left(1 + \frac{m}{2\overline{r}}\right)^2 \overline{r}$$

into the Scharzschild metric to derive its "isotropic" form

$$d\vec{s}^{\,2} = \frac{(1 - m/2\overline{r})^2}{(1 + m/2\overline{r})^2}\, dt^2 - \left(1 + \frac{m}{2\overline{r}}\right)^4 \left(d\overline{x}^2 + d\overline{y}^2 + d\overline{z}^2\right) .$$

Set 11:

(34) Homework, due W 11/29/2017 before class. Shapiro time delay.

(a) Find the Shapiro time delay in the approximation of Rindler p.237 for X_1 the distance from Earth to Sun, X_2 the distance from Mercury to Sun and R the radius of the Sun. State the result in seconds. You may take the parameter values from Wikipedia. Calculate to at least three significant digits.
(b) Estimate the distance light travels in the time period which you find.
(c) Repeat the above estimates with Mercury replaced by Venus.

(35) Homework, due W 11/29/2017 before class. Mercury precession.

(a) Estimate the Einsteinian advance of the perihelion of Mercury in radians from the equation $\triangle \approx 6\pi\, m^2/h^2$, where you may use the approximation $\omega = \sqrt{m/r^3}$ to find the specific angular momentum of Mercury from its frequency.
(b) Find from the previous result the precession of Mercury over 100 earth years and express the result in arc seconds.

(36) Homework, due M 11/27/2017, graduate students only: Prepare to present motion in a central field with potential $U(r) = \alpha/r$ to the class. In particular, derive

$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{r_1}^{r_2} dr \left(E + \frac{\alpha}{r} - \frac{L^2}{2\, m\, r^2}\right)^{-1/2}$$

from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\, m\, \dot{\vec{r}}^{\,2} + \frac{\alpha}{r}\,, \quad \alpha > 0\,.$$

Define and use the effective potential to explain the physically allowed $r(t)$ functions for different values of the enery E

Set 12:

(37) Homework, due M 12/4/2017 before class. Radial coordinate velocity of light in Schwarzschild spacetime. Exercise 11.2 of Rindler.

(38) Homework, due M 12/4/2017 before class. Retarded potential and Lorenz gauge: Show that current conservation $\partial_\mu J^\mu(\{x^\nu\}) = 0$ implies that the retarded solution for the potential [Eq. (7.50) of the book]

$$\Phi^\mu(\{x^\nu\}) = \frac{1}{c}\int d^3y\, \frac{J^\mu(x^0 - |\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}\,,$$

where $y^0 = x^0 - |\vec{x}-\vec{y}|$ is the retarded time, fulfills the Lorenz gauge $\partial_\mu \Phi^\mu(\{x^\nu\}) = 0$.

(39) Homework, due M 12/4/2017 before class. Lorenz gauge with covariant derivatives. Exercise 14.1 of Rindler.

Set 13: Prepare for the Final Thursday, December 14, 3-5 pm.