Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 12

37. Radial coordinate velocity of light in Schwarzschild space-time. Exercise 11.2 of Rindler.

As $\theta,\ \phi = const$,

$$d\vec{s}^{\,2} = \alpha\, dt^2 - \alpha^{-1}\, dr^2 \quad \text{with} \quad \alpha = 1 - 2m/r\,.$$

Light: $d\vec{s}^{\,2} = 0 \ \Rightarrow\ dt = \pm\alpha^{-1}\, dr$. So the coordinate velocity is

$$\frac{dr}{dt} = \pm\alpha \ \to 0 \quad \text{for} \quad r \to 2m\,.$$

For the time to a mirror at r_1 from $r_0 > r_1$ and back, we have

$$\begin{aligned}
\triangle t &= 2\int_{r_1}^{r_0} (1-2m/r)^{-1} dr = 2\int_{r_1}^{r_0} r/(r-2m)\, dr \\
&= 2\int_{r_1}^{r_0} (r-2m+2m)/(r-2m)\, dr = 2\int_{r_1}^{r_0} \left[1 + 2m/(r-2m)\right]\, dr \\
&= 2\left[r_0 - r_1 + 2m\, \ln\left(\frac{r_0-2m}{r_1-2m}\right)\right] \ \to\ \infty \quad \text{for} \quad r_1 \to 2m\,.
\end{aligned}$$

As we watch a source fall towards the horizon, it appears to slow down and to stop at the horizon. By Eq. (11.25) of Rindler the redshift of the light received by us approaches infinity. So the signal becomes ever dimmer, since both the frequency of photons reaching us becomes smaller, and the rate at which they arrive also decreases, each by the Doppler factor.