Special and General Relativity (PHZ 4601/560 Fall 2017) Final December 14, 3-5 pm.

1. **Schwarzschild effective potential (40%).**

   (a) Find the extrema of the Schwarzschild effective potential

   $$V_{GR}(r) = \frac{h^2}{r^2} - \frac{2m}{r} - \frac{2m\,h^2}{r^3}$$

   in the form

   $$r_{1,2} = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}\,,$$

   where you have to calclate p and q in terms of the central mass m and the angular momentum h.

   (b) Choose q so that $(p/2)^2 - q = (p/4)^2$ holds and express h in terms of m for the effective potential resulting from the solution

   $$r_{1,2} = \frac{p}{2} \pm \frac{p}{4}$$

   for its extrema.

   (c) Sketch $V_{GR}(r)$ for the parameters of (b) in the range $0 \le r \le 30m$, $-0.08 < V_{GR} < 0.02$ and compare with Newton's effective potential

   $$V_N(r) = \frac{h^2}{r^2} - \frac{2m}{r}$$

   in the same range.

2. **Schwarzschild radius and density (30%).**

   Consider a sphere of radius R and uniform mass density ρ.

   (a) Find the radius R_0 so that $r_S > R$ holds for $R > R_0$, where r_S is the Schwarzschild radius.

   (b) Calculate R_0 for the density $\rho = 10^{-27}\,[kg/m^3]$, which is a rough approximation for the mass density of the universe. Express the result in light years $[ly]$ an compare with the size of the universe. Use $G = 6.7 \times 10^{-11} m^3 \cdot kg^{-1} \cdot s^{-2}$, $c = 3 \times 10^8\, m/s^2$, and $1\ year = 365 \times 24 \times 60 \times 60\, s$.

Turn over

3. **Einstein's equations (20%).**

   (a) Transform the equation

   $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -\kappa\, T_{\mu\nu}$$

   into the form

   $$R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T \right) .$$

   (b) Consider the Einstein equation with a cosmological constant,

   $$G_{\mu\nu} + g_{\mu\nu}\, \Lambda = -\kappa\, T_{\mu\nu} \, .$$

   Show that $R = n\, \Lambda$ holds in vacuum and find n.

4. **Covariant Lorenz gauge (10%).**

   Prove that $\Phi^{\mu}{}_{;\rho\mu} = -\Phi^{\alpha} R_{\alpha\rho}$ holds in the Lorenz gauge.