Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 8

28. Spaceship using $(d\beta)_S$ and $(d\beta)_E$.

Using unit with $c = 1$ we have $(d\beta)_S = g\, d\tau$ and find from the addition theorem of velocities

$$(d\beta)_E = \frac{\beta + (d\beta)_S}{1 + \beta\,(d\beta)_S} - \beta = \beta\,(1 - \beta\,(d\beta)_S) + (d\beta)_S - \beta = (1 - \beta^2)\,(d\beta)_S\,.$$

After separation of variables and defining $\beta = \beta_E$ and β', $d\beta'$ the corresponding integration variables, we have

$$\int_0^\beta \frac{d\beta'}{1 - \beta'^2} = g \int_0^\tau d\tau'\,.$$

Integrating both sides with Mathematica we find

$$\frac{1}{2}\,\ln\left(\frac{1+\beta}{1-\beta}\right) = g\,\tau \quad \Rightarrow \quad \beta = \tanh(g\tau) \;\text{ and }\; \gamma = \cosh(g\tau)\,.$$

The rest of the problem is solved along the previous lines:

$$\int_0^t dt' = \int_0^\tau \gamma\, d\tau' = \int_0^\tau \cosh(g\tau')\, d\tau'$$

and so on.