Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 4

13. Addition theorem for transverse velocity components.

In K' the motion with velocity $\vec{u'}$ is

$$x'^i \;=\; c^{-1}\,u'^i\,x'^0\,.$$

With respect to frame K the origin of K' frame moves with speed v along the x^1 axis of K. For $i=2,3$ the Lorentz transformations give then

$$x^i \;=\; x'^i \;=\; c^{-1}\,u'^i\,\gamma\,(x^0-\beta\,x^1)\,.$$

Dividing by x^0 gives the velocity components in K:

$$c^{-1}\,u^i = c^{-1}\,u'^i\,\gamma\,(1-\beta\,c^{-1}\,u^1) \;=\; c^{-1}\,u'^i\,\gamma\,\left(1-\beta\,c^{-1}\,\frac{u'^1+v}{1+u'^1v/c^2}\right)\,,$$

where in the second step the already calculated equation for u^1 has been inserted. Bringing everything to the common denominator gives

$$\begin{aligned} u^i &= u'^i\,\gamma\,\frac{(1+\beta\,u'^1/c)-\beta\,c^{-1}\,(u'^1+v)}{1+u'^1v/c^2} \\ &= u'^i\,\gamma\,\frac{1-\beta^2}{1+u'^1v/c^2} \;=\; \frac{u'^i}{\gamma\,(1+u'^1v/c^2)}\,, \end{aligned}$$

where $1-\beta^2=1/\gamma^2$ was used in the last step.