Special and General Relativity (PHZ 4601/5606) Fall 2017 Solutions

Set 5

18. Electromagnetic field tensor in $\vec{E}$ and $\vec{B}$ fields.

A. From the J^4 component we get

$$\partial_4 E^{44}+\partial_1 E^{14}+\partial_2 E^{24}+\partial_3 E^{34} = \frac{4\pi}{c}\, J^4 = 4\pi\,\rho = \nabla\cdot\vec{e} = \partial_1 e^1+\partial_2 e^2+\partial_3 e^3\,.$$

Therefore, $E^{44}=0$ by anti-symmetry, $E^{14}=e^1$, $E^{24}=e^2$ and $E^{34}=e^3$.

From the J^1 component we get

$$\partial_4 E^{41}+\partial_1 E^{11}+\partial_2 E^{21}+\partial_3 E^{31} = \frac{4\pi}{c}\, J^1 = \partial_2 b^3 - \partial_3 b^2 - \partial_4 e^1\,.$$

Therefore, $E^{41}=-e^1$ consistent with $E^{14}=e^1$, $E^{11}=0$ by anti-symmetry, $E^{21}=b^3$ and $E^{31}=-b^2$.

From the J^2 component we get

$$\partial_4 E^{42}+\partial_1 E^{12}+\partial_2 E^{22}+\partial_3 E^{32} = \frac{4\pi}{c}\, J^2 = \partial_3 b^1 - \partial_1 b^3 - \partial_4 e^2\,.$$

Therefore, $F^{42}=-e^2$, $E^{12}=-b^3$ both consistent with anti-symmetry, $E^{22}=0$ by anti-symmetry and (new) $E^{32}=b^1$.

Using anti-symmetry all components are now determined and the $F^{\alpha\beta}$ field tensor reads

$$\left(E^{\alpha\beta}\right) = \begin{pmatrix} 0 & -b^3 & b^2 & e^1 \\ b^3 & 0 & -b^1 & e^2 \\ -b^2 & b^1 & 0 & e^3 \\ -e^1 & -e^2 & -e^3 & 0 \end{pmatrix}\,.$$

The remaining equations from $\beta=3$ are not needed, but can be used for consistency checks.