**Special and General Relativity (PHZ 4601/560 Fall 2017)**

**Solutions Final December 14**

1. **Schwarzschild effective potential.**

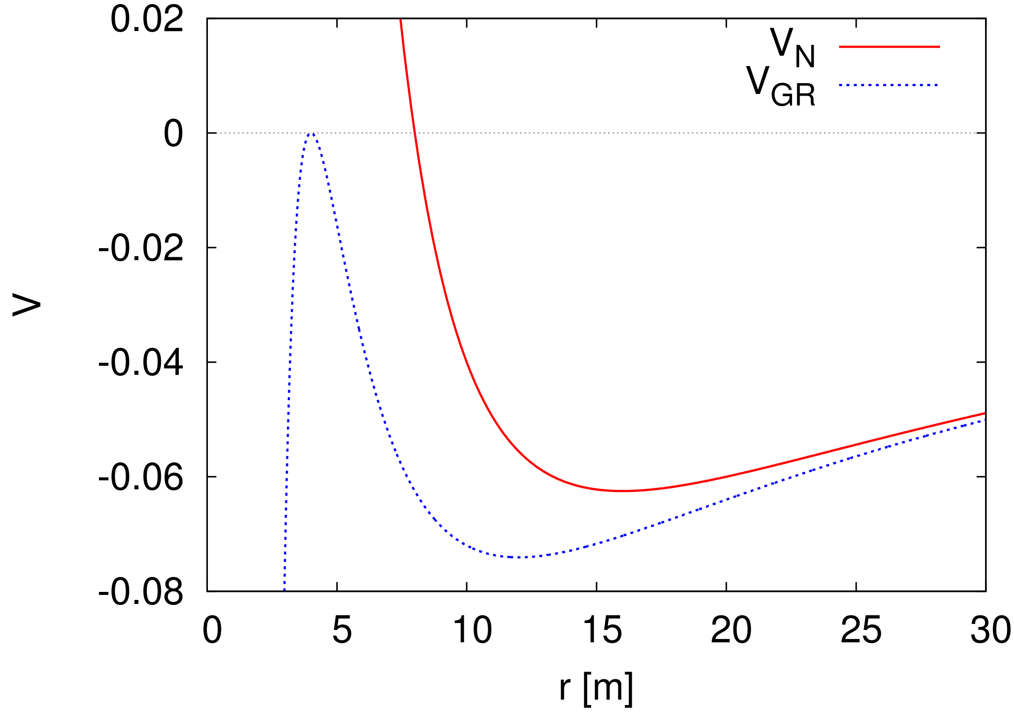


FIG. 1: Newton and Schwarzschild potentials.

(a) We have to calculate the zeros of the derivative of the effective potential

$$V'_{GR}(r) = -2\,\frac{h^2}{r^3} + \frac{2m}{r^2} + \frac{6m\,h^2}{r^4}\,.$$

After multiplying with $r^4$ we arrive at

$$0 = -2\,h^2\,r + 2m\,r^2 + 6m\,h^2 \;\Rightarrow\; 0 = r^2 - \frac{h^2}{m}\,r + 3h^2\,.$$

So, $p = -h^2/m$, $q = 3h^2$ and

$$r_{1,2} = \frac{h^2}{2m} \pm \sqrt{\left(\frac{h^2}{2m}\right) - 3h^2}\,.$$

(b) In terms of $h$ and $m$ the equation $q = (p/2)^2 - (p/4)^2 = 3p^2/16$ reads

$$3h^2 = \frac{3h^4}{16m^2} \ \Rightarrow \ 16m^2 = h^2 \ \Rightarrow \ h = 4m\,,$$

resulting in $r_{1,2} = (8 \pm 4)\,m$ and the Schwarzschild effective potential

$$V_{GR}(r) = \frac{16m^2}{r^2} - \frac{2m}{r} - \frac{32m^3}{r^3}$$

where the first two terms are Newton's potential.

(c) For the chosen parameters the Scharzschild and Newton's potential are drawn in the figure on page 1.

2. **Schwarzschild radius and density.**

(a) The Scharzschild radius is $r_S = 2m$ and

$$m = \frac{4\pi}{3}\, r^3\, \rho\,.$$

Therefore

$$R_0 = \frac{8\pi}{3}\, R_0^3\, \rho \ \Rightarrow \ R_0^2 = \frac{3}{8\pi\,\rho}$$

and $r_S > R$ for $R > R_0$, because $r_s$ increases $\sim R^3$, i.e., faster than $R$.

(b) In conventional units we have $m = G\,M/c^2$ and, therefore,

$$R_0^2 = \frac{3\,c^2}{8\pi\,G\,\rho}\,.$$

With the given numbers, and rounding to three digits, we find $R_0 = 4 \times 10^{27}\,m = 42.3 \times 10^9\,ly$. This is larger than the size of the visible unisverse, which is approximately $15 \times 10^9\,[ly]$.

3. **Einstein's equations.**

(a) We have

$$R^\mu{}_\mu - \frac{1}{2}\, g^\mu{}_\mu\, R = -\kappa\, T^\mu{}_\mu \ \Rightarrow \ R - 2\,R = -\kappa\, T \ \Rightarrow \ R = \kappa\, T$$

and, therefore,

$$R_{\mu\nu} - \kappa\,\frac{1}{2}\, g_{\mu\nu}\, T = -\kappa\, T_{\mu\nu} \ \Rightarrow \ R_{\mu\nu} = -\kappa\,\left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T\right)\,.$$

(b) In vacuum we have

$$R^{\mu}{}_{\mu} - \frac{1}{2}\, g^{\mu}{}_{\mu}\, R + g^{\mu}{}_{\mu}\, \Lambda = 0 \;\Rightarrow\; R - 2\,R + 4\,\Lambda = 0 \;\Rightarrow\; R = 4\,\Lambda\,,$$

i.e., $n = 4$.

4. **Lorenz gauge with covariant derivatives.**

According to (10.55) of Rindler we have

$$\Phi^{\mu}{}_{;\rho\sigma} - \Phi^{\mu}{}_{;\sigma\rho} = -\Phi^{\alpha}\, R^{\mu}{}_{\alpha\rho\sigma}\,.$$

Therefore, the definition (10.68) of the Ricci tensor yields

$$\Phi^{\mu}{}_{;\rho\mu} - \Phi^{\mu}{}_{;\mu\rho} = -\Phi^{\alpha}\, R^{\mu}{}_{\alpha\rho\mu} = -\Phi^{\alpha}\, R_{\alpha\rho}\,.$$

In the Lorenz gauge $\Phi^{\mu}{}_{;\mu\rho} = 0$ holds and we are left with

$$\Phi^{\mu}{}_{;\rho\mu} = -\Phi^{\alpha}\, R_{\alpha\rho}\,.$$